Outline

1. Homogeneous, Anisotropic Media
2. Crystals
3. Plane Waves in Anisotropic Media
4. Wave Propagation in Uniaxial Media
5. Reflection and Transmission at Interfaces
Homogeneous, Anisotropic Media

**Introduction**

- Material equations for homogeneous, anisotropic media

\[
\vec{D} = \epsilon \vec{E} \\
\vec{B} = \mu \vec{H}
\]

- Tensors of rank 2, written as 3 by 3 matrices
  - \(\epsilon\): dielectric tensor
  - \(\mu\): magnetic permeability tensor

- For the following, assume \(\mu = 1\)

- Examples:
  - Crystals, liquid crystals
  - External electric, magnetic fields acting on isotropic materials (glass, fluids, gas)
  - Anisotropic mechanical forces acting on isotropic materials
Properties of Dielectric Tensor

- Maxwell equations imply symmetric dielectric tensor

\[ \epsilon = \epsilon^T = \begin{pmatrix} \epsilon_{11} & \epsilon_{12} & \epsilon_{13} \\ \epsilon_{12} & \epsilon_{22} & \epsilon_{23} \\ \epsilon_{13} & \epsilon_{23} & \epsilon_{33} \end{pmatrix} \]

- symmetric tensor of rank 2 \( \Rightarrow \) coordinate system exists where tensor is diagonal

- orthogonal axes of this coordinate system: principal axes

- elements of diagonal tensor: principal dielectric constants

- 3 principal indices of refraction in coordinate system spanned by principal axes

\[ \vec{D} = \begin{pmatrix} n_x^2 & 0 & 0 \\ 0 & n_y^2 & 0 \\ 0 & 0 & n_z^2 \end{pmatrix} \vec{E} \]

- \( x, y, z \) because principal axes form Cartesian coordinate system
Uniaxial Materials

- isotropic materials: $n_x = n_y = n_z$
- anisotropic materials: $n_x \neq n_y \neq n_z$
- uniaxial materials: $n_x = n_y \neq n_z$
- ordinary index of refraction: $n_o = n_x = n_y$
- extraordinary index of refraction: $n_e = n_z$
- rotation of coordinate system around $z$ has no effect
- most materials used in polarimetry are (almost) uniaxial
Crystal Axes Terminology

- **optic axis** is the axis that has a different index of refraction
- also called *c* or **crystallographic axis**
- **fast axis**: axis with smallest index of refraction
- ray of light going through uniaxial crystal is (generally) split into two rays
- **ordinary ray** (*o-ray*) passes the crystal without any deviation
- **extraordinary ray** (*e-ray*) is deviated at air-crystal interface
- two emerging rays have orthogonal polarization states
- common to use indices of refraction for ordinary ray (*n<sub>o</sub>*) and extraordinary ray (*n<sub>e</sub>*) instead of indices of refraction in crystal coordinate system
- *n<sub>e</sub> < n<sub>o</sub>*: **negative** uniaxial crystal
- *n<sub>e</sub> > n<sub>o</sub>*: **positive** uniaxial crystal
Plane Waves in Anisotropic Media

Displacement and Electric Field Vectors

- plane-wave ansatz for $\vec{D}$, $\vec{E}$, $\vec{H}$

  \[
  \vec{E} = \vec{E}_0 e^{i(\vec{k} \cdot \vec{x} - \omega t)} \\
  \vec{D} = \vec{D}_0 e^{i(\vec{k} \cdot \vec{x} - \omega t)} \\
  \vec{H} = \vec{H}_0 e^{i(\vec{k} \cdot \vec{x} - \omega t)}
  \]

- no net charges in medium ($\nabla \cdot \vec{D} = 0$)

  \[
  \vec{D} \cdot \vec{k} = 0
  \]

  $\vec{D}$ perpendicular to $\vec{k}$

  $\vec{D}$ and $\vec{E}$ not parallel $\Rightarrow$ $\vec{E}$ not perpendicular to $\vec{k}$

  wave normal $\vec{s} = \vec{k}/|\vec{k}|$, energy flow in different directions, at different speeds
Magnetic Field

- Constant, scalar $\mu$, vanishing current density $\Rightarrow \vec{H} \parallel \vec{B}$

- $\nabla \cdot \vec{H} = 0 \Rightarrow \vec{H} \perp \vec{k}$

- $\nabla \times \vec{H} = \frac{1}{c} \frac{\partial \vec{D}}{\partial t} \Rightarrow \vec{H} \perp \vec{D}$

- $\nabla \times \vec{E} = -\frac{\mu}{c} \frac{\partial \vec{H}}{\partial t} \Rightarrow \vec{H} \perp \vec{E}$

- $\vec{D}$, $\vec{E}$, and $\vec{k}$ all in one plane

- $\vec{H}$, $\vec{B}$ perpendicular to that plane

- Poynting vector $\vec{S} = \frac{c}{4\pi} \vec{E} \times \vec{H}$

- Perpendicular to $\vec{E}$ and $\vec{H} \Rightarrow \vec{S}$ (in general) not parallel to $\vec{k}$

- Energy (in general) not transported in direction of wave vector $\vec{k}$
Relation between $\vec{D}$ and $\vec{E}$

- combine Maxwell, material equations in principal coordinate system

\[ D_i = \epsilon_i E_i = n^2 \left( E_i - s_i \left( \vec{E} \cdot \vec{s} \right) \right) \quad i = 1 \ldots 3 \]

- $\vec{s} = \vec{k} / |\vec{k}|$: unit vector in direction of wave vector $\vec{k}$
- $n$: refractive index associated with direction $\vec{s}$, i.e. $n = n(\vec{s})$
- 3 equations for 3 unknowns $E_i$
- eliminate $\vec{E}$ assuming $\vec{E} \neq \vec{0} \Rightarrow$ Fresnel equation

\[
\frac{s_x^2}{n^2 - \epsilon_x} + \frac{s_y^2}{n^2 - \epsilon_y} + \frac{s_z^2}{n^2 - \epsilon_z} = \frac{1}{n^2}
\]

- with $n_i^2 = \epsilon_i$

\[
s_x^2 n_x^2 \left( n^2 - n_y^2 \right) \left( n^2 - n_z^2 \right) + s_y^2 n_y^2 \left( n^2 - n_x^2 \right) \left( n^2 - n_z^2 \right) + s_z^2 n_z^2 \left( n^2 - n_x^2 \right) \left( n^2 - n_y^2 \right) = 0
\]
Electric Field in Anisotropic Material

- Electric field can also be written as

\[ E_k = \frac{n^2 s_k (\vec{E} \cdot \vec{s})}{n^2 - \epsilon_k} \]

- Equivalent to \((a \text{ a constant})\)

\[ \vec{E} = a \begin{pmatrix} s_x \\ \frac{s_x}{n^2 - n_x^2} \\ \frac{s_y}{n^2 - n_y^2} \\ \frac{s_z}{n^2 - n_z^2} \end{pmatrix} \]

- Quadratic equation in \(n\) ⇒ generally two solutions for given direction \(\vec{s}\)

- System of 3 equations can be solved for \(E_k\)

- Denominator vanishes if \(\vec{k}\) parallel to a principal axis ⇒ treat separately
Non-Absorbing, Non-Active, Anisotropic Materials

- $\vec{k}$ not parallel to a principal axis $\Rightarrow$ ratio of 2 electric field components $k$ and $l$

$$\frac{E_k}{E_l} = \frac{s_k (n^2 - \epsilon_l)}{s_l (n^2 - \epsilon_k)}$$

- Ratio is independent of electric field components
- $n^2$ and $\epsilon$ real $\Rightarrow$ ratios are real $\Rightarrow$ electric field is linearly polarized
- In non-absorbing, non-active, anisotropic material, 2 waves propagate that have different linear polarization states and different directions of energy flows
- Direction of vibration of $\vec{D}$ corresponding to 2 solutions are orthogonal to each other (without proof)

$$\vec{D}_1 \cdot \vec{D}_2 = 0$$
Introduction

- uniaxial media $\Rightarrow$ dielectric constants:

\[
\begin{align*}
\epsilon_x &= \epsilon_y = n_o^2 \\
\epsilon_z &= n_e^2
\end{align*}
\]

- second form of Fresnel equation reduces to

\[
\left( n^2 - n_o^2 \right) \left[ n_o^2 \left( s_x^2 + s_y^2 \right) \left( n^2 - n_e^2 \right) + s_z^2 n_e^2 \left( n^2 - n_o^2 \right) \right] = 0
\]

- two solutions $n_1$, $n_2$ given by

\[
\begin{align*}
n_1^2 &= n_o^2 \\
1 \quad &= \quad s_x^2 \quad + \quad s_y^2 \quad + \quad \frac{s_z^2}{n_e^2} \quad + \quad \frac{s_z^2}{n_o^2}
\end{align*}
\]
Propagation in General Direction

- (unit) wave vector direction in spherical coordinates

\[
\hat{s} = \begin{pmatrix}
  s_x \\
  s_y \\
  s_z
\end{pmatrix} = \begin{pmatrix}
  \sin \theta \cos \phi \\
  \sin \theta \sin \phi \\
  \cos \theta
\end{pmatrix}
\]

- \(\theta\): angle between wave vector and optic axis
- \(\phi\): azimuth angle in plane perpendicular to optic axis

\[
\frac{1}{n_o^2} = \cos^2 \theta + \frac{\sin^2 \theta}{n_e^2}
\]

\[
n_2(\theta) = \sqrt{\frac{n_o n_e}{\sqrt{n_o^2 \sin^2 \theta + n_e^2 \cos^2 \theta}}}
\]

- take positive root, negative value corresponds to waves propagating in opposite direction

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Ordinary and Extraordinary Rays

- from before

\[
\frac{1}{n_2^2} = \frac{\cos^2 \theta}{n_o^2} + \frac{\sin^2 \theta}{n_e^2}
\]

\[
n_2(\theta) = \frac{n_o n_e}{\sqrt{n_o^2 \sin^2 \theta + n_e^2 \cos^2 \theta}}
\]

- \(n_2\) varies between \(n_o\) for \(\theta = 0\) and \(n_e\) for \(\theta = 90^\circ\)

- first solution propagates according to ordinary index of refraction, independent of direction ⇒ ordinary beam or ray

- second solution corresponds to extraordinary beam or ray

- index of refraction of extraordinary beam is (in general) not the extraordinary index of refraction
Ordinary Beam

- ordinary beam speed independent of wave vector direction
- for \( D_i = \epsilon_i \tilde{E}_i = n^2 \left( \tilde{E}_i - s_i (\tilde{E} \cdot \tilde{s}) \right) \), \( i = 1 \cdots 3 \) to hold for any direction \( \tilde{s} \), \( \tilde{E}_o \cdot \tilde{s} = 0 \) and \( E_{o,z} = 0 \)

- electric field vector of ordinary beam (with real constant \( a_o \neq 0 \))

\[
\tilde{E}_o = a_o \begin{pmatrix} \sin \phi \\ - \cos \phi \\ 0 \end{pmatrix}
\]

- ordinary beam is linearly polarized
- \( \tilde{E}_o \) perpendicular to plane formed by wave vector \( \tilde{k} \) and \( c \)-axis
- displacement vector \( \tilde{D}_o = n_o \tilde{E}_o \parallel \tilde{E}_o \)
- Poynting vector \( \tilde{S}_o \parallel \tilde{k} \)
Extraordinary Ray

- since $\vec{D}_e \cdot \vec{k} = 0$ and $\vec{D}_e \cdot \vec{D}_o = 0 \Rightarrow$ unique solution (up to real constant $a_e$)

\[
\vec{D}_e = a_e \begin{pmatrix} \cos \theta \cos \phi \\ \cos \theta \sin \phi \\ -\sin \theta \end{pmatrix}
\]

- since $E_e \cdot D_o = 0$, $D_e = \epsilon \vec{E}_e$

\[
\vec{E}_e = a \begin{pmatrix} n_e^2 \cos \theta \cos \phi \\ n_e^2 \cos \theta \sin \phi \\ -n_o^2 \sin \theta \end{pmatrix}
\]

- uniaxial medium $\Rightarrow \vec{E}_o \cdot \vec{E}_e = 0$

- however, $\vec{E}_e \cdot \vec{k} \neq 0$
Dispersion Angle

- angle between $\vec{k}$ and Poynting vector $\vec{S} = \text{angle between } \vec{E}$ and $\vec{D} = \text{dispersion angle}$

$$\tan \alpha = \frac{\left| \vec{E}_e \times \vec{D}_e \right|}{\vec{E}_e \cdot \vec{D}_e} = \frac{(n_e^2 - n_o^2) \tan \theta}{n_e^2 + n_o^2 \tan^2 \theta} = \frac{\sin 2\theta}{2} \frac{(n_e^2 - n_o^2)}{n_o^2 \sin^2 \theta + n_e^2 \cos^2 \theta}$$

- equivalent expression

$$\alpha = \theta - \arctan \left( \frac{n_o^2}{n_e^2} \tan \theta \right)$$

- for given $\vec{k}$ in principal axis system, $\alpha$ fully determines direction of energy propagation in uniaxial medium

- for $\theta$ approaching $\pi/2$, $\alpha = 0$

- for $\theta = 0$, $\alpha = 0$
Propagagation Direction of Extraordinary Beam

- angle $\theta'$ between Poynting vector $\vec{S}$ and optic axis
  \[
  \tan \theta' = \frac{n_o^2}{n_e^2} \tan \theta
  \]

- ordinary and extraordinary wave do (in general) not travel at the same speed

- phase difference in radians between the two waves given by
  \[
  \frac{\omega}{c} \left( n_2(\theta) d_e - n_o d_o \right)
  \]

- $d_{o,e}$: geometrical distances traveled by ordinary and extraordinary rays
Propagation Along c Axis

- plane wave propagating along c-axis $\Rightarrow \theta = 0$
- ordinary and extraordinary beams propagate at same speed $\frac{c}{n_0}$
- electric field vectors are perpendicular to c-axis and only depend on azimuth $\phi$
- ordinary and extraordinary rays are indistinguishable
- uniaxial medium behaves like an isotropic medium
- example: “c-cut” sapphire windows
Propagation Perpendicular to c Axis

- plane wave propagating perpendicular to c-axis $\Rightarrow \theta = \pi/2$

$$\vec{E}_o = \begin{pmatrix} \sin \phi \\ -\cos \phi \\ 0 \end{pmatrix}$$

- $\vec{E}_o$ perpendicular to plane formed by $\vec{k}$ and c-axis
- electric field vector of extraordinary wave

$$\vec{E}_e = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

- $\vec{E}_e$ parallel to c-axis
- direction of energy propagation of extraordinary wave parallel to $\vec{k}$ since $\vec{E}_e \parallel \vec{D}_e$
Phase Delay between Ordinary and Extraordinary Rays

- ordinary and extraordinary wave propagate in same direction
- ordinary ray propagates with speed \( \frac{c}{n_0} \)
- extraordinary beam propagates at different speed \( \frac{c}{n_e} \)
- \( \vec{E}_o, \vec{E}_e \) perpendicular to each other \( \Rightarrow \) plane wave with arbitrary polarization can be (coherently) decomposed into components parallel to \( \vec{E}_o \) and \( \vec{E}_e \)
- 2 components will travel at different speeds
- (coherently) superposing 2 components after distance \( d \) \( \Rightarrow \) phase difference between 2 components \( \frac{\omega}{c} (n_e - n_o) d \) radians
- phase difference \( \Rightarrow \) change in polarization state
- basis for constructing linear retarders
Summary: Wave Propagation in Uniaxial Media

- ordinary ray propagates like in an isotropic medium with index $n_o$
- extraordinary ray sees direction-dependent index of refraction

$$n_2(\theta) = \frac{n_0 n_e}{\sqrt{n_0^2 \sin^2 \theta + n_e^2 \cos^2 \theta}}$$

- $n_2$: direction-dependent index of refraction of the extraordinary ray
- $n_0$: ordinary index of refraction
- $n_e$: extraordinary index of refraction
- $\theta$: angle between extraordinary wave vector and optic axis

- extraordinary ray is not parallel to its wave vector
- angle between the two is dispersion angle

$$\tan \alpha = \frac{(n_e^2 - n_o^2) \tan \theta}{n_e^2 + n_o^2 \tan^2 \theta}$$
General case

- from isotropic medium \((n_i)\) into uniaxial medium \((n_o, n_e)\)
- \(\theta_i\): angle between surface normal and \(\vec{k}_i\) for incoming beam
- \(\theta_{1,2}\): angles between surface normal and wave vectors of (refracted) ordinary wave \(\vec{k}_1\) and extraordinary wave \(\vec{k}_2\)
- phase matching at interface requires

\[
\vec{k}_i \cdot \vec{x} = \vec{k}_1 \cdot \vec{x} = \vec{k}_2 \cdot \vec{x}
\]

- \(\vec{x}\): position vector of a point on interface surface

\[
n_i \sin \theta_i = n_1 \sin \theta_1 = n_2 \sin \theta_2
\]

- \(n_1 = n_o\): index of refraction of ordinary wave
- \(n_2\): index of refraction of extraordinary wave
Ordinary and Extraordinary Rays

- ordinary wave $\Rightarrow$ Snell’s law

$$\sin \theta_1 = \frac{n_l}{n_1} \sin \theta_l$$

- law for extraordinary ray not trivial

$$n_l \sin \theta_l = n_2 (\theta(\theta_2)) \sin \theta_2$$

- (in general) $\theta_2$ and therefore $\vec{k}_2$ will not determine direction of extraordinary beam since Poynting vector (in general) not parallel to wave vector

- solve for $\theta_2 \Rightarrow$ determine direction of Poynting vector

- special cases reduce complexity of equations
Extraordinary Ray Refraction for General Case

\[ \cot \theta_2 = n_0 \frac{\sqrt{n_0^2 n_e^2 + n_e^2 c_x^2 (n_e^2 - n_o^2)} - n_o^2 - (n_e^2 - n_o^2) (c_x^2 + c_y^2)}{n_o^2 + c_x^2 (n_e^2 - n_o^2)} \]

propagation vector of extraordinary ray

\[ S_x = \cos \alpha \cos \theta_2 + \frac{\sin \alpha \sin \theta_2 (c_x \sin \theta_2 - c_y \cos \theta_2)}{\sqrt{c_z^2 + (c_x \sin \theta_2 - c_y \cos \theta_2)^2}} \]

\[ S_y = \cos \alpha \sin \theta_2 - \frac{\sin \alpha \cos \theta_2 (c_x \sin \theta_2 - c_y \cos \theta_2)}{\sqrt{c_z^2 + (c_x \sin \theta_2 - c_y \cos \theta_2)^2}} \]

\[ S_z = c_z \ast \frac{\sin \alpha}{\sqrt{c_z^2 + (c_x \sin \theta_2 - c_y \cos \theta_2)^2}} \]

\( \vec{c} \) optic axis vector \( \vec{c} = (c_x, c_y, c_z)^T \)

\( \vec{S} \) propagation direction of extraordinary ray \( \vec{S} = (S_x, S_y, S_z)^T \)

\( \theta_1 \) angle between \( \vec{k}_I \) and interface normal

\( \theta_2 \) angle between \( \vec{k}_e \) and interface normal

\( \alpha \) dispersion angle
Normal Incidence

- normal incidence $\Rightarrow \theta_I = 0, \theta_1 = \theta_2 = 0$
- choose plane formed by surface normal and crystal axis
- both wave vectors and ordinary ray not refracted
- extraordinary ray refracted by dispersion angle $\alpha$

$$\alpha = \theta - \arctan \left( \frac{n_o^2}{n_e^2} \tan \theta \right)$$
Optic Axis in Plane of Incidence and Plane of Interface

- \( \theta + \theta_2 = \pi/2 \Rightarrow \cot \theta_2 = \frac{n_e}{n_o} \cot \theta_1 \)
- \( \theta_1 \): angle between surface normal and *ordinary* ray or wave vector
  \( \sin \theta_I = n_o \sin \theta_1 \)
- extraordinary wave sees equivalent refractive index
  \[
  n_y = \sqrt{n_e^2 + \sin^2 \theta_I \left(1 - \frac{n_e^2}{n_o^2}\right)}
  \]
- direction of Poynting vector
  \[
  S_x = \cos(\theta_2 + \alpha) \\
  S_y = \sin(\theta_2 + \alpha) \\
  S_z = 0
  \]
- determine dispersion angle \( \alpha \) and *add* to \( \theta_2 \) to obtain direction of extraordinary ray
Optic Axis Perpendicular to Plane of Incidence

- $c$-axis perpendicular to plane of incidence $\Rightarrow \theta = \frac{\pi}{2}$, $n_2 \left(\frac{\pi}{2}\right) = n_e$

$$n_l \sin \theta_l = n_e \sin \theta_2$$

- extraordinary wave vector obeys Snell’s law with index $n_e$
- $\theta = \frac{\pi}{2} \Rightarrow$ dispersion angle $\alpha = 0$
- Poynting vector $\parallel$ wave vector, extraordinary beam itself obeys Snell’s law with $n_e$
- double refraction only for non-normal incidence
ordinary ray follows Snell’s law
transmitted extraordinary wave vector and ray coincide
exit of extraordinary wave on interface defined by extraordinary ray
extraordinary wave vector follows Snell’s law with index $n_2(\theta)$

\[ n_I \sin \theta_E = n_2 \sin \theta_U \]

- $n_I$ index of isotropic medium
- $\theta_E$ angle of wave/ray vector with surface normal in isotropic medium
- $n_2, \theta_U$ corresponding values for extraordinary wave vector in uniaxial medium
- $n_2$ is function of $\theta$ normally already known from beam propagation in uniaxial medium
- $\theta_U$ is function of geometry of interface,
- plane-parallel slab of uniaxial medium, $\theta_E = \theta_I$, (in general) extraordinary beam displaced on exit
Total Internal Reflection (TIR)

- TIR also in anisotropic media
- \( n_o \neq n_e \Rightarrow \) one beam may be totally reflected while other is transmitted
- principal of most crystal polarizers
- example: calcite prism, normal incidence, optic axis parallel to first interface, exit face inclined by 40°
- \( \Rightarrow \) extraordinary ray not refracted, two rays propagate according to indices \( n_o, n_e \)
- at second interface rays (and wave vectors) at 40° to surface
- 632.8 nm: \( n_o = 1.6558, n_e = 1.4852 \)
- requirement for total reflection \( \frac{n_u}{n_l} \sin \theta_U > 1 \)
- with \( n_l = 1 \Rightarrow \) extraordinary ray transmitted, ordinary ray undergoes TIR