Lecture 6: Fixed Retarders

Outline

- Jones and Mueller Matrices for Linear Retarders
- Zero and Multiple Order Linear Retarders
- Crystal Retarders
- Polymer Retarders
- Achromatic Retarders
- Angle-Dependence of Linear Retarders
- Temperature Dependence of Fixed Retarders
- Spectral Fringes in Retarders
- Linear Retarder Selection Guide

Introduction

- retarder: splits beam into 2 components, retards phase of one component, combines components at exit into single beam
- ideal retarder does not change intensity of light or degree of polarization
- any retarder is characterized by two (not identical, not trivial) Stokes vectors of incoming light that are not changed by retarder
- eigenvectors of retarder
- depending on polarization described by eigenvectors, retarder is linear retarder, circular retarder, elliptical retarder
- linear, circular retarders are special cases of elliptical retarders
- circular retarders sometimes called rotators
- linear retarders by far the most common type of retarder

Jones Matrix for Linear Retarders

- linear retarder with fast axis at 0° characterized by Jones matrix

\[
J_r(\delta) = \begin{pmatrix} e^{i\delta} & 0 \\ 0 & 1 \end{pmatrix}, \quad J_r(\delta) = \begin{pmatrix} e^{i\delta/2} & 0 \\ 0 & e^{-i\delta/2} \end{pmatrix}
\]

- \(\delta\): phase shift between two linear polarization components (in radians)
- absolute phase does not matter ⇒ ‘symmetric’ version avoids absolute phase that depends on retardation
- use of ‘asymmetric’ version led to some erroneous theoretical calculations of instrumental Mueller matrix of telescopes

Mueller Matrices for Linear Retarders

- corresponding Mueller matrix is given by

\[
M_r = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos\delta & -\sin\delta \\ 0 & 0 & \sin\delta & \cos\delta \end{pmatrix}
\]

- linear retarder, fast axis angle \(\theta\), retardance \(\delta\)

\[
\begin{pmatrix} 1 & 0 \\ 0 & \cos^2 2\theta + \cos\delta \sin^2 2\theta \\ 0 & \cos 2\theta \sin 2\theta - \cos\delta \cos 2\theta \sin 2\theta \\ 0 & -\sin 2\theta \sin \delta \end{pmatrix}
\]

- 2 or more linear retarders in series ⇒ (in general) equivalent to 1 elliptical retarder
Retarders on the Poincaré Sphere

- Retarder eigenvector (fast axis) in Poincaré sphere
- Points on sphere are rotated around retarder axis by amount of retardation

Zero and Multiple-Order Linear Retarders

- Most retarders based on birefringent materials
- Typical birefringent materials: quartz, mica, and polymer films
- c-axis parallel to interface
- Retardation (delay between ordinary and extraordinary ray):
  \[ N\lambda = d(n_e - n_o) \]
  - \( d \): geometrical thickness
  - \( \lambda \): wavelength
  - \( n_e, n_o \): indices of refraction for extraordinary and ordinary rays
  - \( N \): retardation expressed in waves
- Quarter wave plate is obtained with \( N = m + \frac{1}{4} \) and \( m \) being an integer
  - \( m = 0 \): true zero-order retarder
  - \( m > 0 \): multi-order retarder

Wavelength Dependence of Retarders

- Wavelength dependence of 2-mm multi-order and true zero-order quartz quarter-wave retarder assuming constant \( n_o \) and \( n_e \)
  - Retardation of 1.25 waves equivalent 0.25 waves
  - The larger \( d \), the faster retardation changes as function of wavelength

Quartz Retarders

- Quartz available in large sizes
- Can be produced artificially
- Most commonly used crystal for high-quality retarders
- True zero-order quartz-wave retarder in visible: 15 \( \mu \)m thick
- Very difficult to manufacture
- Compound zero-order retarder: two \( \sim 1 \)-mm thick plates with difference in thickness corresponding to desired zero-order retarder
  - Plates optically contacted with fast axes at 90°
  - Plates cancel each other except for small path-length difference
  - Usable from about 180 nm to 2700 nm
Mica Retarders
- natural mica often used for commercial retarders
- cheap, available in large sizes (20 cm by 20 cm)
- mica crystals easily cleaved into very thin sheets
- quarter-wave plate in visible ~50 µm thick
- transparent from 350 nm to 6 µm, but absorbs even in visible
- thicker at longer wavelengths ⇒ larger absorption

Polymer Retarders
- stretched polymers (e.g. polyvinyl alcohol) also birefringent
- fast axis perpendicular to stretch direction
- quarter-wave retarder is ≈20 µm in visible
- true zero-order retarders
- highly transparent even in UV
- sizes up to 40 cm

Achromatic Retarders

Different Birefringent Materials
- retarders highly wavelength sensitive due to
  - wavelength itself
  - wavelength dependence of the birefringence
- combine two materials with opposite variations of $\Delta n = n_e - n_o$ with wavelength
- choose appropriate thicknesses for achromatic retarder (perfect retardance at 2 wavelengths)

\[
N\lambda_1 = d_a\Delta n_{1a} + d_b\Delta n_{1b} \\
N\lambda_2 = d_a\Delta n_{2a} + d_b\Delta n_{2b}
\]

- $N$: desired retardance
- $\lambda_1$, $\lambda_2$: two wavelengths where correct retardation is achieved
- $d_a$, $d_b$: thicknesses of plates made from materials $a$ and $b$
- $\Delta n_{ij}$: birefringence for material $j$ at wavelength $i$

Bicrystalline Retarders
- solve equations for two thicknesses

\[
d_a = N\frac{\lambda_1\Delta n_{2b} - \lambda_2\Delta n_{1b}}{\Delta n_{1a}\Delta n_{2b} - \Delta n_{1b}\Delta n_{2a}} \\
d_b = N\frac{\lambda_2\Delta n_{1a} - \lambda_1\Delta n_{2a}}{\Delta n_{1a}\Delta n_{2b} - \Delta n_{1b}\Delta n_{2a}}
\]

- achromatic retarder as long as denominator $\neq 0$
- negative thickness $d_{a,b}$ ⇒ fast axis at 90°
- if thickness too small ⇒ replace with compound retarder
- better: numerically optimize over desired wavelength range
- quartz and MgF$_2$ used most often in visible
- different materials have widely different off-axis performance
- combined temperature dependence also material dependent
- trade off wavelength versus field-of-view versus temperature variation of retardance

Combinations of Retarders of the Same Material
- several retarders made from same material (Pancharatnam 1955)
- half-wave plate: outer two plates parallel fast axes, inner plate rotated by $\approx 60°$
- fast axis direction of combined retarder depends on wavelength
- also achromatic quarter-wave plates, but not as good

Theoretical variation of retardation and fast axis orientation as a function of relative wavelength for a Pancharatnam achromatic half-wave plate
Superachromatic Retarders

- 3 bicrystalline retarders in Pancharatnam configuration
- fast axis direction depends on wavelength
- angular acceptance angle very limited for crystal achromats
- much better angular performance with plastic retarders

Fresnel Rhomb Performance

- BK7 at 632.8 nm; retardance at 55.08°
- total internal reflection on glass ($n_i$) air interface for $n_i \sin \beta > 1$
- $\beta$: (internal) angle of incidence, phase shift $\delta$
- $\tan \delta/2 = -\frac{\cos \beta \sqrt{n_i^2 \sin^2 \beta - 1}}{n_i \sin^2 \beta}$
- retardation strongly depends on angle ⇒ small acceptance angle
- variation of retardance with wavelength purely due to variation of index of refraction with wavelength

Fresnel Rhombs

- traditional arrangements for quarter-wave (left) and half-wave (right)
- Fresnel rhombs
- phase shift on total internal reflection (TIR) on interface between dielectrics
- in the visible: not possible to achieve 90° phase shift on single reflection
- several reflections can produce $\lambda/4$ and $\lambda/2$ retardation

Overview of Wavelength-Dependence
Angle-Dependence of Birefringent Retarders

- retardation by uniaxial medium depends on angle of incidence
- retardation changes because index of extraordinary ray depends on direction
- apparent plate thickness changes for both rays
- for $\sin^2 \theta < n_o^2, n_e^2$

\[
\delta \approx \delta(\theta = 0) \left[ 1 + \frac{\sin^2 \theta}{2n_o} \left( \frac{n_e^2}{n_o} - \frac{\cos^2 \phi}{n_o} \right) \right]
\]

- $\theta$ angle of incidence
- $\phi$ angle between plane of incidence and optic axis
- $\delta(\theta = 0)$ retardation at normal incidence

Temperature Dependence of Fixed Retarders

- retardation depends on temperature
- optical path difference variation in linear approximation

\[
\delta T d (n_e - n_o) + d (\delta T n_e - \delta T n_o)
\]

- $\delta T$ indicates variations with temperature
- with coefficient of thermal expansion (CTE) $\alpha = \delta d / d$

\[
\delta N = N \left( \alpha + \frac{\delta n_e - \delta n_o}{n_e - n_o} \right)
\]

- quartz: $\delta N = N \left( -1.0 \times 10^{-4} \right) \text{ K}^{-1}$ at 632.8 nm
- 2 mm thick $\lambda/4$ retarder $\Rightarrow$ retardation variation $\approx 1^\circ$ per Kelvin
- compound zero-order same as true zero-order retarders
- achromatic retarders made from different materials show stronger temperature dependence

Angle-Dependence of Birefringent Retarders (continued)

\[
\delta \approx \delta(\theta = 0) \left[ 1 + \frac{\sin^2 \theta}{2n_o} \left( \frac{n_e^2}{n_o} - \frac{\cos^2 \phi}{n_o} \right) \right]
\]

- $\delta$ decreases when optic axis in plane of incidence ($\phi = 0$)
- $\delta$ increases when optic axis perpendicular to plane of incidence ($\phi = \pi/2$)
- linear retarder with slightly wrong retardance can be tipped or tilted to achieve required retardance
- retardation error proportional to retardance $\Rightarrow$ multiple-order waveplates much worse than zero-order retarders
- compound zero-order retarders also much worse performance
- second retarder has retardation with opposite sign
- but retardation error also has opposite sign because azimuth changes by 90°
- retardation error increases quadratically with angle and is inversely proportional to index squared

Spectral Fringes in Retarders

For true zero-order quartz retarders using Berreman calculus

- retarders are not ideal because of interference between reflected and transmitted beams at 2 interfaces (Fabry-Perot)
- optical thickness is different for ordinary and extraordinary beam
- retardation (and transmittance) show spectral fringes
Comparison of various types of commercially available zero-order retarders; quartz and \( \text{MgF}_2 \) are compound zero-order retarders; accuracy in percent refers to half-wave plate.

<table>
<thead>
<tr>
<th>Type</th>
<th>Accuracy (%)</th>
<th>Wavelength Range (nm)</th>
<th>Bandpass (nm)</th>
<th>Acceptance Angle (°)</th>
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<td>Quartz</td>
<td>0.4</td>
<td>180-2700</td>
<td>100</td>
<td>3</td>
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<tr>
<td>( \text{MgF}_2 )</td>
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<td>140-6200</td>
<td>100</td>
<td>3</td>
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<td>Mica</td>
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<td>240-2000 330-1000</td>
<td>330-1000</td>
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