1. Symmetries and the Form of the Lagrangian

In the formulation of the minimum principle that leads to the equations of motion, the Lagrange density could, in principle, be anything. Its actual form is dictated by a symmetry.

One might wonder whether one must simply guess at the form of the Lagrangian, such as the choice v^2, or if there is a more basic and compelling way for finding a useful expression.

There may be many ways, but it seems for the moment that the most sweeping and general way for adapting our minimum principle to the facts of Nature is by way of a *symmetry*. That is to say, we note that "certain things remain the same when we change certain other things," and derive a Lagrangian which embodies that regularity with which Nature presents us.

The remarkable thing is, that certain symmetries are so powerful and so restrictive that they entirely determine the functional form of the Lagrangian, and therefore the equations of motion. The beginnings of this concept go back a long way. In one of Huygens' publications on the motion of bodies he shows a ball dropping from the top of the mast of a boat, straight down the mast. He then asks: how does someone standing on shore, seeing the boat glide by, see the path of that ball?

The very least we can say about our world is: **the Universe is made of particles, space and time**. Note that this implies that space and time must be seen on an equal footing with particles; that is to say, they are not some sort of invisible graph paper, but real stuff, the cement between the particles. As such, space and time may have a structure, just as matter does. Matter is an arrangement of particles; spacetime is an arrangement of points $(\vec{x}, t)$.

Having noted that Nature is built of particles, space and time, we must of course ask about their interrelationship: *where is what when?* We are thus interested in the positions of particles, and the change of position in the course of time. In other words, we try to describe the Lagrangian as a function of position $\vec{x}$, time t, and velocity $\vec{v} \equiv d\vec{x}/dt$ (we will see that the observed symmetries of Nature make it unnecessary to consider higher derivatives such as $d^2\vec{x}/dt^2$). The action is then

$$S = \int \mathcal{L}\, dt = \int \mathcal{L}(x_i, v_i, t)\, dt \tag{1.1}$$

in which I have written the position and velocity in component notation. The variation along the path is

$$\delta\mathcal{L} = \frac{\partial\mathcal{L}}{\partial x_i}\epsilon + \frac{\partial\mathcal{L}}{\partial v_i}\frac{d\epsilon}{dt} \tag{1.2}$$

Partial integration of the second term produces

$$\delta S = \frac{\partial\mathcal{L}}{\partial v_i}\epsilon\bigg|_A^B - \int \left(\frac{d}{dt}\frac{\partial\mathcal{L}}{\partial v_i} - \frac{\partial\mathcal{L}}{\partial x_i}\right)\epsilon\, dt = 0 \tag{1.3}$$

Because, as before, our minimum principle requires $\delta S = 0$, and ϵ is arbitrary between its zero end points, the equation of motion is

$$\frac{d}{dt}\frac{\partial\mathcal{L}}{\partial v_i} - \frac{\partial\mathcal{L}}{\partial x_i} = 0 \tag{1.4}$$ ♠

The choice of a minimum principle as the basis of a theory of dynamics immediately implies that the Lagrangian has certain properties. First, minimising a function means that we get the same equations of motion if we multiply that function with an arbitrary constant. Second, because it is an integral which we minimise, we get the same motion if we add to the integrand the total time derivative of an arbitrary function of space-time:

$$\mathcal{L}^* \equiv \mathcal{L} + \frac{df(\vec{x}, t)}{dt} \tag{1.5}$$

because then

$$S^* = \int \mathcal{L}^* dt = \int \mathcal{L}\, dt + \int \frac{df(\vec{x},t)}{dt}\, dt = S + f_B - f_A \tag{1.6}$$

The additional terms are fixed, so that they have no influence on the equations of motion.

The above two properties of $\mathcal{L}$ are mathematical consequences of our decision to use a minimum principle. But further restrictions on $\mathcal{L}$ must be built in by using clues from Nature. These may be of any kind, but in practice the most powerful and general have proven to be *symmetries.*

To begin with, consider the Lagrangian $\mathcal{L}_0$ of a free particle. The first symmetry is that of the `isotropy of space`. That is to say, the motion of a free particle does not depend on the orientation of our coordinate system. We conclude immediately that this means that $\mathcal{L}_0$ cannot depend on the direction of $\vec{v}$, but must be a function

$$\mathcal{L}_0 = \mathcal{L}_0(v^2, \vec{x}, t) \tag{1.7}$$

The second symmetry is the `homogeneity of space and time`. That is to say, the motion of a free particle does not depend on the place from which we measure its progress, nor does it depend on the time at which we start our measurement. Accordingly, $\mathcal{L}_0$ cannot explicitly depend on $\vec{x}$ or t, so that

$$\mathcal{L}_0 = \mathcal{L}_0(v^2) \tag{1.8}$$

The forceful influence of these symmetries becomes apparent when we derive the equation of motion from this $\mathcal{L}_0$:

$$\frac{d}{dt}\frac{\partial \mathcal{L}_0}{\partial v_i} - \frac{\partial \mathcal{L}_0}{\partial x_i} = \frac{d}{dt}\frac{\partial \mathcal{L}_0}{\partial v_i} = 0 \tag{1.9}$$

from which we immediately conclude that *a free particle moves with constant speed.* Thus, we find that the Law of Inertia follows from our minimum principle if we take the two basic symmetries of spacetime into account.

In parentheses, I note that the discovery of symmetry involves a lot of thought. In our everyday environment particles do not behave like Kipling's cat, namely as if all places and all times are alike to them. The deduction that space-time is homogeneous and isotropic involves considerable idealization and perspicacity (which is of course why it took so long to get to that point!)

The above symmetries pertain to rotation and translation of our coordinates. But it so happens that there is yet another symmetry to be observed in Nature: the motion of a free particle does not depend on a steady translation of our coordinate system. That is to say, if we observe a free particle, at rest in a given coordinate system, from a vantage point that moves with velocity $\vec{w}$ with respect to this system, we observe that the particle is still free, i.e. it moves with constant velocity, but with velocity $-\vec{w}$.

This remarkable fact of Nature is called `Galilei-Huygens relativity`, or `Galilei-Huygens symmetry`. The homogeneity and isotropy of space-time have already served to derive the form in Eq.(1.8) for the Lagrangian of the free particle; adding G-H symmetry determines $\mathcal{L}_0$ completely. To demonstrate this, suppose that a particle moves with velocity $\vec{v}$, and that it is observed from a coordinate system moving with a velocity $\vec{w}$ that is small compared with $\vec{v}$. In that second system the Lagrangian is

$$\mathcal{L}_0^* = \mathcal{L}_0((\vec{v}+\vec{w})^2) \tag{1.10}$$

To first order in w this can be written as

$$\mathcal{L}_0^* \simeq \mathcal{L}_0(v^2) + 2\vec{v}\cdot\vec{w}\frac{\partial \mathcal{L}_0}{\partial v^2} \tag{1.11}$$

The additional term must lead to the same particle motion. We saw above that only the addition of a term that is a total time derivative leaves the motion unchanged. The extra term is proportional to $\vec{v}$, which is a total time derivative already; so we conclude that we must have

$$\frac{\partial \mathcal{L}_0}{\partial v^2} = \text{constant} \tag{1.12}$$

In other words, *the Lagrangian of a free particle is a scalar multiple of the square of the velocity.* This conclusion allows us to write down the Lagrangian after a Galilei-Huygens transformation with finite velocity $\vec{w}$:

$$\mathcal{L}_0^* = (\vec{v} + \vec{w})^2 = v^2 + 2\vec{v}\cdot\vec{w} + w^2 = v^2 + \frac{d}{dt}\left(2\vec{x}\cdot\vec{w} + t\,w^2\right) \tag{1.13}$$

The additional terms have been written explicitly as a total time derivative which, as we saw above, has no influence on the equations of motion.

This completes the derivation of $\mathcal{L}_0$. Apparently we cannot tolerate any more restrictions in the form of additional symmetries. In particular, Nature tells us that steady rotation does *not* leave the motion invariant. This has always been a source of wonder, from Newton with his rotating bucket of water to Mach with his 'motion with respect to the distant stars.'

Recalling the arguments given above, one may argue that there is one more freedom of the Lagrangian which we have not yet disposed of, namely the possibility of multiplying it with a scalar factor. Leaving aside trivialities, such as a global change of physical units, we see that the freedom is no longer absolute when one considers the motion of *two* free particles, each with its own Lagrangian $\mathcal{L}_1$ and $\mathcal{L}_2$. The simultaneous motion of the particles is found from

$$\mathcal{L} = \mathcal{L}_1 + \mathcal{L}_2 \tag{1.14}$$

which implies that only the *ratio* of the pre-factors is free. We may use one particle in the whole Universe as a standard and scale the rest accordingly, but this amounts to a trivial choice of physical units. Therefore we must conclude that each particle has its own Lagrangian pre-factor. For reasons that become apparent when we consider interacting particles, this factor is written as $m/2$, so that the free Lagrangian finally takes the form

$$\mathcal{L}_0 = \frac{1}{2}mv^2 \tag{1.15}$$

It is clear that this pre-factor must not be negative, or else the Lagrange formalism wouldn't produce the required minimum in the action.

The motion of a non-free particle may be described in many ways, depending on the nature of the influence acting on it. An easily recognizable case is obtained when we try the simplest generalization of the Lagrangian, namely subtracting a function Φ from it:

$$\mathcal{L} = \frac{1}{2}mv^2 - \Phi(\vec{x}) \tag{1.16}$$ ♠

Substitution into the Lagrangian equation gives

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial v_i} - \frac{\partial \mathcal{L}}{\partial x_i} = \frac{d\,m\vec{v}}{dt} + \vec{\nabla}\Phi = 0 \tag{1.17}$$

which is the classical equation of motion if we identify Φ as an external potential (and which, incidentally, justifies *a posteriori* our pre-factor $\frac{1}{2}m$).

As a concluding remark I note that the above mechanism of using symmetries to nail down the functional form of $\mathcal{L}$ works wonderfully in other cases, too. In fact, all known forces of Nature can be derived from a symmetry principle which, remarkably enough, not only prescribes the Lagrangian of a free particle (like the way in which we obtained v^2 above), but *the interactions as well.*