I. Introduction

1.1 Classical and contemporary mechanics.

It seems to me that there are several reasons why students often find cosmology and general relativity very forbidding subjects. Among these reasons are:

(1) there appears to be no connection with familiar subjects like classical mechanics, and even relativistic mechanics;

(2) the subject is introduced via a very complicated series of concepts from differential geometry. The formulations are highly arcane for a physicist, and yet by no means rigorous enough for a mathematician;

(3) it is made to appear as if gravitation is a force that is dramatically and forever different from the other forces of Nature.

I will try to remedy all these, by presenting an approach that is never used in the traditional textbooks, but that nevertheless connects directly to our contemporary understanding of forces. Namely, I will present the gravitational field as being due to a local gauge symmetry (what this means should become clear in the sequel). In order to do this, I will go back to basics, and first discuss classical mechanics and the Lagrangian formalism, stressing the role of symmetries in determining the motion of mechanical systems. Second, I will introduce relativity via Lorentz symmetry, and point out that this symmetry (and indeed all others!) cannot be valid globally throughout spacetime, because the speed of light is finite and maximal. Third, it will be shown that it is possible to have symmetries that differ from event to event in space-time, provided that there be (for each symmetry) an additional field, mediating a force. Fourth, that the field generated by local Lorentz symmetry is the gravitational field, and that the corresponding field equation is the Einstein equation.

Back to classical mechanics, then. The question we are trying to answer is: "where is what when?" As we proceed in discussing this question, we will see a gradual process of "unification" (I can't think of any nicer expression, unfortunately): in Newtonian mechanics, three totally unrelated ingredients are needed, namely time for "when", space for "where", and mass for "what." Then, in special relativistic mechanics, we observe that time and space cannot be unrelated, and are not some sort of invisible graph paper on which our Universe is plotted, but instead form a continuum called spacetime. Finally, it is found that mass and force are also connected to spacetime, namely via its symmetries. (The full story requires the addition of the quantum rules, but that is as yet impossible for the gravitational field; I'll give some references to basic papers on quantum symmetries).

The "where" and "when" are quantitatively described by dynamical variables; the position $\vec{x}$ and time t, and the rate of change of $\vec{x}$, the velocity $\vec{v} = d\vec{x}/dt$, form a set of variables of which $\vec{x}$ and $\vec{v}$ are dynamical. Note, by the way, that only position and time differences are observable; likewise, only velocity diferences are observable.

In general, $\vec{x} = \vec{x}(t)$, so one can find $\vec{v} = d\vec{x}/dt$;

then one should expect $\vec{v} = \vec{v}(t)$, and one can define $\vec{a} = d\vec{v}/dt$. Should one continue with $d\vec{a}/dt$, etc.? This is not necessary, because one observes that $\vec{a} = 0$ unless a force acts.

Can the effects of forces be deduced from the force free case? No, but we can get ideas about how to do it, from consideration of the free particle. On a free particle, by definition $\vec{a} = 0$ so that $\vec{v} =$ const. and the only dynamical variables describing it are $\vec{x}$ and $\vec{v}$.

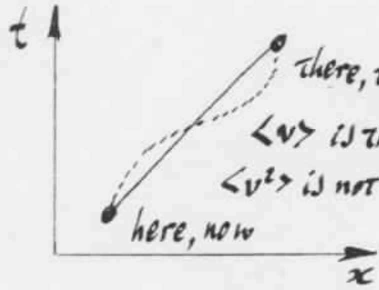
Can we devise a function of $\vec{x}$ and $\vec{v}$ that summarizes the motion of a free particle? Yes, if we use the observed symmetries of space and time. Namely, (1) space is homogeneous, i.e. there is translation invariance ("one point in space is as good as another"), (2) time is homogeneous ("one point in time is as good as another"); note that this is about the same as saying that only spatial and temporal differences are observable: the Universe is not drawn on some kind of absolute graph paper!

These homogeneities mean that $\vec{x}$ and t cannot occur explicitly in the function that describes the motion of a free particle.

Then we have the symmetry of (3) space is isotropic ("one direction in space is as good as another"), which implies that the function that describes a free particle cannot contain any directional information → no vectors, only scalars; the only variable that is not yet excluded, namely $\vec{v}$ (remember $\vec{x}$ and t were dropped because of translational symmetry in space & time) can therefore only occur as a scalar, i.e. v^2.

Thus, there can at best be a function (called the Lagrangian L) that summarizes the motion of a free particle, that is a function of the speed (i.e. the magnitude of the velocity) only: $\mathcal{L} = \mathcal{L}(v^2)$.

How can the motion of the particle be deduced from $\mathcal{L}$? Let's take a particle with a given average speed $\langle v \rangle$, and – for simplicity – in one dimension. In a spacetime plot, keeping $\langle v \rangle$ fixed means that the end points of the path are fixed. But the path itself could be anything. Is there any path that is in any way distinguished from the others? Yes: a straight line through the two fixed points.



Since $\mathcal{L} = \mathcal{L}(v^2)$ by the symmetries of space and time, one observes directly that the average of $\mathcal{L}$ over any path is a <u>minimum</u> for that straight line: because v^2 is positive definite, "the average over the square is never smaller than the square of the average", and $\int v^2 dt$ is minimal if v is a constant.

If we require that $\int \mathcal{L}\, dt$ is minimal, we get that a free particle moves at constant speed in any coordinate direction. This is observed, so that it seems that the demand that $\mathcal{L}$ fulfil this extremal property leads to the correct description of the motion.

We can now <u>suppose</u> that the requirement that $\int \mathcal{L} dt$ is extremal, also describes a non-free particle.

The question that arises then, is: what Lagrangian can we use? That is, what functional combination of the dynamical variables describes the motion? (this is a more sophisticated restatement of "where is what when"). The homogeneity of space & time, and the isotropy of space, allows only v^2 for the free particle; and the influence of "the outside world" (which *may* depend on $\bar{x}$ and t) come into $\mathcal{L}$ as an additive term (notice that the above construction, in which $\mathcal{L}$ is a function of which the time integral must be extremal, requires that $\mathcal{L}$ be *additive*, so we can *add* "outside world" terms (sometimes called force or potential energy terms) to the $\mathcal{L}$ of the free particle.

The additional "force" terms can be limited by using *symmetries*, just as the free-particle $\mathcal{L}$ was narrowed down to $\mathcal{L}(v^2)$. Note, by the way, that adding a force term allows $\mathcal{L}$ to vary with time; $\mathcal{L}$ is not constant → itself not good for describing motion, but we can use a *property* of $\mathcal{L}$ to describe motion. As said above, the property to be used is that $\int \mathcal{L}\,dt$ should be extremal.

1.2 Lorentz symmetry

In what follows, it will be our strategy to restrict the additional "force" terms in the Lagrangian as much as possible. It will turn out that such a restriction can be made very severe by demanding symmetric behaviour of the extra terms, especially if the symmetry has *group* character.

To get some idea how symmetries occur and can be used, consider the isotropy of space (rotational invariance). If space is symmetric under rotations, one should expect there to be a characteristic invariant, and indeed there is: rotations leave the *Pythagoras* distance $D^2 = x^2 + y^2 + z^2$ unchanged.

Now it is observed (Michelson-Morley expt.) that the speed of light is invariant, i.e. a light signal that propagates, in a coordinate system K, according to $x = \pm ct$, propagates in another K' according to $x' = \pm ct'$ with the same c. Thus, $x^2 = c^2t^2$ and $x'^2 = c^2t'^2$, and one guesses that

$s^2 \equiv c^2t^2 - x^2 =$ invariant. This is the Minkowski distance, and (in analogy with the Pythagoras recipe) we may ask: if s^2 is invariant, what is the corresponding symmetry? From a comparison with the Pythagoras distance, it is evident that it will be an analogue of rotation, i.e. a linear transformation from K to K' with determinant positive. It turns out to be the Lorentz symmetry; in matrix form in one dimension, it is

$$L\vec{x} = \gamma \begin{pmatrix} 1 & -v \\ -v/c^2 & 1 \end{pmatrix} \begin{pmatrix} x \\ t \end{pmatrix} = \vec{x}' \quad (1.2.1)$$

in which $$\gamma \equiv (1 - v^2/c^2)^{-1/2} \quad (1.2.2)$$

Note that it is rather silly to keep writing c all the time instead of requiring $c = 1$, but in the astronomical literature c is usually written explicitly.

Now we can try to construct the Lorentz symmetric Lagrangian of a free particle. It must be a Lorentz scalar, just as in the preceding paragraph $\mathcal{L}$ had to be a "Galilei scalar", i.e. $\mathcal{L} = \mathcal{L}(v^2)$. In the above we see, that the only available scalar is s^2; and just as we used $\int \mathcal{L}\, dt$ for the classical "action", we will use the average $\int ds$ in the Lorentz symmetric case:

$$\mathcal{S} = -\alpha \int ds \qquad (1.2.3)$$

where the constant $-\alpha$ is to be determined by correspondence with classical mechanics. In that case,

$$\mathcal{S} = \int \mathcal{L}\, dt \qquad (1.2.4)$$

The Lorentz transformation of dt allows us to connect it with ds:

$$dt' = dt \left(1 - \frac{dx^2 + dy^2 + dz^2}{c^2 dt^2}\right)^{1/2} \rightarrow dt' = \frac{ds}{c} = \frac{1}{\gamma} dt \qquad (1.2.5)$$

Therefore, $\mathcal{L} = -\alpha c/\gamma$; the classical limit $v \rightarrow 0$ gives $\alpha = mc$ (verify yourself), so that

$$\mathcal{L} = -mc^2/\gamma \qquad (1.2.6)$$

for a free relativistic particle (notice the family resemblance with $E = \gamma mc^2$!).

1.3 Local symmetry

In what follows, we will try to restrict the additional "force" terms in $\mathcal{L}$, which must occur for non-free particles, by introducing symmetries. As a matter of notation, we will call the dynamical variables Q and $Q_{,\mu}$, where symbolically

$$Q_{,\mu} \equiv \partial Q / \partial x_\mu \qquad (1.3.1)$$

These are analogous to (x, v) (undergraduate mechanics), $(q, \dot{q})$ (Hamiltonian mechanics), $(\psi, \partial\psi)$ (quantum mechanics), or $(\gamma\psi, \gamma\,\partial\psi)$ (Dirac relativistic quantum mechanics).

Because the most general case we'll need is likely to appear buried under indices, we will first go through the whole program in the simplest case, namely a scalar symmetry; this will also be a good way to introduce the technique of finding a functional dependence of $\mathcal{L}$, i.e. we seek an answer of the form $\mathcal{L} = \mathcal{L}(\ldots, \ldots, \ldots)$ rather than an arithmetic answer like $\mathcal{L} = 8\pi$.

Suppose, then, that we have a symmetry operator $\mathbb{L}$, for which there is experimental evidence (e.g. Lorentz symmetry, charge-exchange symmetry, or whatever). We can then ask: can we construct a Lagrangian that is symmetric under a global application of $\mathbb{L}$?

The answer is <u>no</u>. That might seem upsetting, but let us trace its origin. If we were to try and apply $\mathbb{L}$ everywhere in the Universe, we would find that once one had specified what symmetry to apply (e.g. "a rotation over 23 degrees"), one would not be free to do anything different elsewhere (e.g. 47 degrees). But this is not consistent with relativity (i.e. Lorentz symmetry), because <u>the speed of light is finite and maximal</u>. After all, if we applied $\mathbb{L}(23°)$ here, how would the folks at Arcturus know? They'd have to wait several years until they knew that it wasn't supposed to be $\mathbb{L}(47°)$.

Thus, <u>global</u> symmetries ought not to occur, but yet we do not want to drop the idea of symmetry altogether, because there is so much evidence for it! There is a way out, namely admitting <u>local</u> symmetry, i.e. some $\mathbb{I}$ that is not fixed everywhere (like $\mathbb{I}(23°)$, say) but that varies from point to point: $\mathbb{I}(23°)$ here, $\mathbb{I}(47°)$ at Arcturus, in general $\mathbb{I}(x_\mu)$, where x_μ is an event in spacetime.

But this, at first, seems impossible. For consider the origin of the equations of motion. We had required that the action

$$S = \int \mathcal{L}\, dx_\mu \qquad (1.3.2)$$

be an extremum, i.e. if we apply an infinitesimal variation to $\mathcal{L}$, the variation in S should be zero,

$$\delta S = \delta \int \mathcal{L}\, dx_\mu = \int \delta\mathcal{L}\, dx_\mu = 0 \qquad (1.3.3)$$

If $\mathcal{L}$ is symmetric under some $\mathbb{I}$, we still have

$$\delta S = \int \delta \mathcal{L}\, dx_\mu = \int \delta\, \mathbb{I}\mathcal{L}'\, dx_\mu = \int \mathbb{I}\, \delta\mathcal{L}'\, dx = 0 \quad (1.3.4)$$

provided, of course, that $\mathbb{I}$ is the same everywhere; if not, there would be an additional integrand $\delta\mathbb{I}\ \mathcal{L}'$ in (1.3.4)! Thus, <u>a local symmetry must be expected to spoil the equations of motion</u>.

The way out is, to add terms to $\mathcal{L}$ that restore $\delta S = 0$. This is extremely important because, as we had seen, <u>extra terms in $\mathcal{L}$ represent forces</u>. Thus, <u>a local symmetry generates a force</u>!

Consequently, our programme is the following:
(1) Find some kind of symmetry $\mathbb{I}$ and require that the Lagrangian be symmetric under local $\mathbb{I}_l$.
(2) This is only possible if $\mathcal{L}$ contains extra terms to counteract the fact that $\delta \mathbb{I}_l \neq 0$, due to the local character of $\mathbb{I}_l$.
(3) These extra terms correspond to (a) force field(s).
(4) See if any such forces can be identified with those that are observed to occur.

It will turn out that local Lorentz symmetry generates a force that can be identified with gravity.

The main features of the approach in which a force is derived from a local symmetry (often called a "gauge symmetry") are the following:

* local symmetry of $\mathcal{L}$ → introduce field to patch things up;
* mathematical form in which the patching-up field occurs in $\mathcal{L}$ depends on the symmetry; if it's a group, then certain spatial derivatives & combinations thereof come in, so that the group character is preserved
* group behaviour enters via the structure constants (these are a measure of the non-commutative behaviour of the group)
* non-commutative symmetries produce nonlinear field combinations in $\mathcal{L}$, because the free field can couple to itself
* the structure constants and the spatial derivatives together can be written as a new form of differentiation, the covariant derivative $\mathcal{D}$.

II. Fields from local symmetry.

2.1. Scalar symmetry

Suppose the symmetry of $\mathcal{L}$ is simply scalar multiplication. Let $\mathcal{L} = \mathcal{L}(Q, Q_{,\mu})$ be the Lagrangian with dynamical variables Q and $Q_{,\mu}$, and the action is

$$S = \int_{\Omega} \mathcal{L}\, d^4x_\mu \qquad (2.1.1)$$

integrated over an arbitrary domain Ω. If $\delta S = 0$ for an arbitrary infinitesimal perturbation, it follows immediately that $\delta\mathcal{L} = 0$, because Ω is arbitrary. Therefore,

$$\delta\mathcal{L} = \frac{\partial\mathcal{L}}{\partial Q}\,\delta Q + \frac{\partial\mathcal{L}}{\partial Q_{,\mu}}\,\delta Q_{,\mu} = 0 \qquad (2.1.2)$$

and the Lagrangian equations of motion hold. Note that we are allowed here to subject $\mathcal{L}$ to the same symmetry under which Q is supposed to be symmetric, because $\mathcal{L} = \mathcal{L}(Q, Q_{,\mu})$.

Now let Q be changed infinitesimally by some local symmetry I_1. If I is simply scalar multiplication, we can write

$$Q \rightsquigarrow Q + \delta Q; \quad \delta Q = \varepsilon(x)\, Q \qquad (2.1.3)$$

It follows immediately that, if $Q_{,\mu} \rightsquigarrow Q_{,\mu} + \delta Q_{,\mu}$,

$$\delta Q_{,\mu} = \varepsilon_{,\mu}\, Q + \varepsilon(x)\, Q_{,\mu} \qquad (2.1.4)$$

so that the variation of the action contains

$$\delta\mathcal{L} = \left\{ \frac{\partial \mathcal{L}}{\partial Q} Q + \frac{\partial \mathcal{L}}{\partial Q_{,\mu}} Q_{,\mu} \right\} \varepsilon + \frac{\partial \mathcal{L}}{\partial Q_{,\mu}} \varepsilon_{,\mu} Q \qquad (2.1.5)$$

The term { } drops out because of the equations of motion (2.1.2), and evidently $\delta\mathcal{L} \neq 0$ because of the term $\varepsilon_{,\mu}$: the local character of the transformation spoils the proper extremum behaviour of $\mathcal{L}$, and no good equations of motion result!

In other words, the locality of $\mathbb{I}$ produces a mismatch between the $\mathbb{I}\mathcal{L}$ at one point and the $\mathbb{I}\mathcal{L}$ elsewhere. This mismatch can possibly be patched up by adding terms to $\mathcal{L}$. Suppose that there is such a "patch-up" field; let it be called A'. Then we get a new Lagrangian

$$\mathcal{L}' = \mathcal{L}'(Q, Q_{,\mu}, A') \qquad (2.1.6)$$

of which we require that $\delta\mathcal{L}' = 0$. This prescribes a functional dependence of $\mathcal{L}'$ on its arguments, as follows. The δQ and $\delta Q_{,\mu}$ come from (2.1.3, 4) as before; the variation $\delta A'$ is, of course, a linear combination of ε and $\varepsilon_{,\mu}$, as long as the transformation is infinitesimal. The most general form for $\delta A'$ is then

$$\delta A' = U\,\varepsilon(x)\,A' + C^{\mu}\,\varepsilon_{,\mu} \qquad (2.1.7)$$

with constant scalar U and vector C^{μ}, to be determined afterwards. We require

$$\delta\mathcal{L}' = 0 = \frac{\partial \mathcal{L}'}{\partial Q}\delta Q + \frac{\partial \mathcal{L}'}{\partial Q_{,\mu}}\delta Q_{,\mu} + \frac{\partial \mathcal{L}'}{\partial A'}\delta A' \qquad (2.1.8)$$

Inserting the expressions of δQ etc. gives a linear equation in ε and $\varepsilon_{,\mu}$. Because the magnitude of ε is arbitrary (as long as it's small), each coefficient of ε and $\varepsilon_{,\mu}$ must vanish independently. This gives

$$\frac{\partial \mathcal{L}'}{\partial Q} Q + \frac{\partial \mathcal{L}'}{\partial Q_{,\mu}} Q_{,\mu} + \frac{\partial \mathcal{L}'}{\partial A'} \mathcal{U} A' = 0 \qquad (2.1.9)$$

$$\frac{\partial \mathcal{L}'}{\partial Q_{,\mu}} Q + \frac{\partial \mathcal{L}'}{\partial A'} C^{\mu} = 0 \qquad (C^{\mu} C_{\mu} = 1) \qquad (2.1.10)$$

$$\frac{\partial \mathcal{L}'}{\partial Q_{,\mu}} Q + \frac{\partial \mathcal{L}'}{\partial A_{\mu}} = 0 \qquad (2.1.11)$$

which follows from (2.1.10) with the definition of the vector field A_μ by

$$A_\mu \equiv C_\mu A' \qquad (2.1.12)$$

Note that if C^{μ} didn't have an inverse, then at least two of the equations (2.1.10) would be linearly dependent, and the system could not be solved.

The equation (2.1.11) is a prescription of the way in which $\mathcal{L}'$ must depend on Q, $Q_{,\mu}$, and A_μ. We see directly that it requires that the vector field A_μ, introduced to patch up the mismatch created by the locality of I_1 (i.e. the x-dependence of ε), occur in $\mathcal{L}'$ <u>only</u> through the combination

$$Q_{,\mu} - Q A_\mu \qquad (2.1.13)$$

If we define the <u>covariant derivative</u> by

$$Q_{;\mu} \equiv Q_{,\mu} - Q A_\mu \qquad (2.1.14)$$

we see that the form $\mathcal{L} = \mathcal{L}(Q, Q_{,\mu}, A')$ allows us only <u>one</u> way to insert $Q_{;\mu}$ into $\mathcal{L}'$, namely to make it depend on $Q_{;\mu}$ in the same way as $\mathcal{L}$ depends on $Q_{,\mu}$. This must be so because $Q_{;\mu}$ contains a term that is linear in $Q_{,\mu}$, and another term that can be made zero by choosing $\mathbf{I} = \mathbf{II}$. Thus,

$$\mathcal{L}' = \mathcal{L}(Q, Q_{;\mu}) \qquad (2.1.15)$$

and from now on we use this form.

<u>Note</u> in the covariant derivative (2.1.14), that the local symmetry prescribes that Q and A_μ couple in the product QA_μ; those who know quantum electrodynamics will recognize the covariant derivative in the form where the dynamical variables $Q_{,\mu}$ and Q are replaced by the derivative $\partial/\partial x$ and a constant factor ie:

$$Q_{,\mu} - QA_\mu \rightsquigarrow \partial - ieA \rightsquigarrow i\psi^*(\gamma.\partial - ie\gamma\cdot A)\psi \qquad (2.1.16)$$

which is the famous "minimal coupling" term in the Dirac equation (the γ's are the Dirac matrices).

Having found that the form of the Lagrangian is determined by (2.1.14, 15), it remains to determine the constants $\mathcal{U}$ and C^μ. First, we note that

$$\delta A_\mu = C_\mu \, \delta A' = C_\mu \, \mathcal{U} \, \varepsilon(x) A' + \varepsilon_{,\mu} = C_\mu C^\nu \mathcal{U} \, \varepsilon A_\nu + \varepsilon_{,\mu} \qquad (2.1.17)$$

Second, we recall the expressions for $\delta\mathcal{L}$ and $\delta\mathcal{L}'$ from (2.1.5, 8), which lead directly to

$$\frac{\partial \mathcal{L}'}{\partial Q} = \frac{\partial \mathcal{L}}{\partial Q}\Big|_{Q_{;\mu}} - \frac{\partial \mathcal{L}}{\partial Q_{;\mu}}\Big|_{Q} \varepsilon A_\mu \qquad (2.1.18)$$

$$\frac{\partial \mathcal{L}'}{\partial Q_{,\mu}} = \frac{\partial \mathcal{L}}{\partial Q_{;\mu}}\Big|_{Q} \qquad (2.1.19)$$

$$\frac{\partial \mathcal{L}'}{\partial A'} = -\frac{\partial \mathcal{L}}{\partial Q_{;\nu}}\Big|_{Q} \varepsilon C_\nu Q \qquad (2.1.20)$$

These can be inserted into (2.1.9), which gives

$$\left\{\frac{\partial \mathcal{L}}{\partial Q} Q + \frac{\partial \mathcal{L}}{\partial Q_{;\mu}} Q_{;\mu}\right\} - \frac{\partial \mathcal{L}}{\partial Q_{;\nu}} \varepsilon Q \mathcal{U} A_\nu = 0 \qquad (2.1.21)$$

The term {} vanishes because of (2.1.15), and because this $\mathcal{L}$ must obey the equations of motion (2.1.2); it follows immediately that

$$\mathcal{U} = 0 \qquad (2.1.22)$$

The definition of A_μ then gives, by means of the expression for $\delta A'$,

$$\delta A_\mu = \varepsilon_{,\mu} \qquad (2.1.23)$$

This demonstrates clearly how the vector field A_μ comes in because of the <u>local</u> character of the symmetry; if $\mathbb{I}_1$ were global, $\varepsilon_{,\mu} = 0$!

We have found how the combination (2.1.14) of Q, $Q_{,\mu}$ and A_μ can occur in $\mathcal{L}$, but having introduced A_μ to make up for the local character of $\mathbb{L}$ we are now obliged to allow A_μ to occur in $\mathcal{L}$ as a free field. Can the locality of $\mathbb{L}$ restrict these free fields also?

The patch-up vector field A_μ may occur itself in $\mathcal{L}$ as a dynamical variable, together with its spatial derivative $A_{\mu,\nu}$, just like Q and $Q_{,\mu}$. Since $\mathcal{L}$ is additive, we can first determine the Lagrangian $\mathcal{L}_0$ that contains the A's only, and then add it to the previous $\mathcal{L}$. We use the same variational form:

$$\delta\mathcal{L}_0 = \frac{\partial\mathcal{L}_0}{\partial A_\mu}\delta A_\mu + \frac{\partial\mathcal{L}_0}{\partial A_{\mu,\nu}}\delta A_{\mu,\nu} \qquad (2.1.24)$$

As usual, we insert δA_μ, and require that each coefficient of ε, $\varepsilon_{,\mu}$, etc. vanish independently. This gives the two equations

$$\frac{\partial\mathcal{L}_0}{\partial A_\mu} = 0 \qquad (2.1.25)$$

$$\frac{\partial\mathcal{L}_0}{\partial A_{\mu,\nu}} + \frac{\partial\mathcal{L}_0}{\partial A_{\nu,\mu}} = 0 \qquad (2.1.26)$$

Accordingly, we find that A_μ itself cannot occur in the Lagrangian. Consequently, the patch-up field is not an observable; but it can couple to the dynamical variable Q, as evident in (2.1.14). Then (2.1.26) shows that A can occur in the Lagrangian only through the combination

$$F_{\mu\nu} = A_{\nu,\mu} - A_{\mu,\nu} \qquad (2.1.27)$$

Thus, the curl of the vector field is observable! Note: this should look very familiar if you know Maxwell's Equations.

This completes our demonstration of the fact that the requirement of local symmetry of $\mathcal{L}$ is so severe, that even the way in which A_μ can couple to the other fields and in which it can occur itself in $\mathcal{L}$ is prescribed entirely! We will see below that this holds for vector fields Q^A also. In that case, it will become clear that the cleanness with which $\mathcal{L}$ is prescribed is due to the group structure of the symmetry. Other symmetries might be usable, but would most likely be messy and complicated.

Note, in the above, that the A-field occurs only linearly: terms of the form $A_\mu A_\nu$ or $A_\mu A_{\nu,\kappa}$ do not appear in $\mathcal{L}$ under the local $\mathbb{I}$ symmetry.

For each symmetry, there is a conservation law (Noether's Theorem). The conserved quantity associated with the above scalar-multiplication symmetry can be found by considering the dependence of the total Lagrangian $\mathcal{L}_T \equiv \mathcal{L}_0 + \mathcal{L}$ on the field A_μ; this can be shown formally by considering $\delta\mathcal{L}_T$, but it is plausible immediately on the grounds that the A_μ were introduced directly by the symmetry $\mathbb{I}$. The quantity

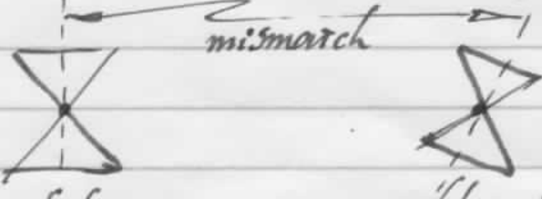
$$J^\mu \equiv \partial\mathcal{L}_T / \partial A_\mu \qquad (2.1.28)$$

is a <u>current</u>, and obeys the conservation law

$$J^\mu_{,\mu} = 0 \qquad (2.1.29)$$

From global to local Lorentz symmetry

Local Lorentz symmetry basically means "I can always choose my coordinates to be Lorentz symmetric, but the neighbours may not agree".



Note the difference between the intrinsic properties of space time, and the extrinsic properties (coordinates)

my light cone   neighbour's light cone

The mismatch shows up as a complication in the coordinates: from $\eta_{\mu\nu}$ to a general bilinear form. Or, physically, by a drifting apart of free-float lines (= appearance of a force).

$$d^2\tau = d^2t - d^2x - d^2y - d^2z \qquad d^2\tau' = d^2t' - d^2x' - d^2y' - d^2z' \qquad (54)$$

my interval   neighbour's interval

mismatch: mapping $x_\alpha \leftrightarrow x'_\alpha$

(See Weinberg eq. 3.2.5 – 7)

Suppose that the neighbours and I agree on the use of a common coordinate system ξ^α. Then I can perform the mapping

$$dx^\mu = \frac{\partial x^\mu}{\partial \xi^\alpha}\, d\xi^\alpha \qquad \text{or} \qquad d\xi^\alpha = \frac{\partial \xi^\alpha}{\partial x^\mu}\, dx^\mu \qquad (55)$$

Accordingly, the interval equation becomes

$$d^2\tau = g_{\mu\nu}\, dx^\mu dx^\nu \tag{56}$$

$$g_{\mu\nu} \equiv \frac{\partial \xi^\alpha}{\partial x^\mu} \frac{\partial \xi^\beta}{\partial x^\nu} \eta_{\alpha\beta} \tag{57}$$

In other words, $g_{\mu\nu}$ is the distance recipe which the neighbours and I agree on; I can always choose it in such a way that $g_{\mu\nu}$ (which is a function of x^μ!) coincides with $\eta_{\mu\nu}$ at my location.

The distance recipe expressed above is a little more involved than the globally Lorentzian $\eta_{\mu\nu} = (+---)$, but it is still recognisable as a distance in space time. Thus, $g_{\mu\nu} \equiv$ <u>metric tensor</u>

→ Distance recipe = curvature ! ←

The case of the Anatolian clay tablets

Because I can choose $g_{\mu\nu} = \eta_{\mu\nu}$ where I'm standing, I can also choose

$$u^\mu = \frac{dx^\mu}{d\tau} = \frac{\partial x^\mu}{\partial \xi^\alpha} v^\alpha \tag{58}$$

← <u>local</u> Lorentz free-float !

From globally free to locally free

In a globally Lorentz symmetric world, the four-momentum of a particle is rigorously conserved. That is to say,

$$\frac{dp^{\mu}}{d\tau} = 0 \qquad (59)$$

The substantial derivative $\frac{d}{d\tau}$ can be written, as usual, as a sum over partial derivatives, by using the Lorentz interval $d^2\tau = d^2t - d^2x$, in the form customary in implicit functions (see e.g. Courant & Hilbert), so that

$$\frac{d}{d\tau} = u^{\alpha} \frac{\partial}{\partial x^{\alpha}} \qquad (60)$$

Now we go from a globally free motion to a locally free motion (cf. what Wheeler calls "free float"). This entails a change:

$$(61)$$

$\frac{\partial}{\partial x^{\sigma}}$ in global Lorentz symm. $\rightsquigarrow$ $\frac{\partial}{\partial x^{\sigma}}$ + "mismatch terms" in local Lorentz symm.

See Weinberg Eqs. (4.6.4), (5.1.6), (5.1.7):

$$\frac{du^{\alpha}}{d\tau} = \frac{d^2\xi^{\alpha}}{d\tau^2} = 0 \quad \text{in free-float} \qquad (62)$$

$\Big\downarrow$ from ξ to x

$$0 = \frac{d}{d\tau}\left(\frac{\partial \xi^{\alpha}}{\partial x^{\mu}} \frac{dx^{\mu}}{d\tau}\right) = \frac{\partial \xi^{\alpha}}{\partial x^{\mu}} \frac{d^2x^{\mu}}{d\tau^2} + \frac{\partial^2 \xi^{\alpha}}{\partial x^{\mu}\partial x^{\nu}} \frac{dx^{\mu}}{d\tau} \frac{dx^{\nu}}{d\tau} \qquad (63)$$

Upon multiplication with $\partial x^\lambda/\partial\xi^\alpha$ we get

$$\frac{d^2x^\lambda}{d\tau^2} + \Gamma^{\lambda}_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} = 0 \qquad (64)$$

because $\frac{\partial\xi^\alpha}{\partial x^\mu}\frac{\partial x^\lambda}{\partial\xi^\alpha} = \delta^\lambda_\mu$; or,

$$\frac{du^\lambda}{d\tau} + \Gamma^{\lambda}_{\mu\nu}u^\mu u^\nu = 0 \equiv \frac{Du^\lambda}{D\tau} \qquad (65)$$

$$\underbrace{\Gamma^{\lambda}_{\mu\nu} \equiv \frac{\partial x^\lambda}{\partial\xi^\alpha}\frac{\partial^2\xi^\alpha}{\partial x^\mu\partial x^\nu}} \qquad (66)$$

extent of the mismatch due to local Lorentz symmetry || Weinberg eqs. (3.2.8 - 3.2.9)

Thus, the equation of motion $Du^\lambda/D\tau = 0$ is the same as $du^\lambda/d\tau = 0$: the particle is still free, but we use curvilinear coordinates!

In a local gauge theory, the force disappears as an external entity.

What connects matter and distance?

The distance recipe $g_{\mu\nu}$ causes matter to move, if $g_{\mu\nu} \neq \eta_{\mu\nu}$. But this shifting of matter then contributes to a change of $g_{\mu\nu}$ itself.

In order to incorporate this new physics we need some observational guidance; pure thought will do us no good! We try to extrapolate from the Newtonian case, where the equation of motion is

$$\frac{d\vec{u}_i}{dt} = -\frac{\partial \Phi}{\partial x^i} \qquad (67)$$

In the Newtonian case, we have <u>nonrelativistic</u> particles, so (1) we can neglect $dx^i/d\tau$ compared with $dt/d\tau$, and (2) we get that $dt/d\tau$ is a constant (from the expression for $d^2\tau = g_{00}\, d^2t + \dots$). Hence the equation of motion becomes, in a time-independent g,

$$\frac{du^\mu}{d\tau} + \Gamma^\mu_{00} \left(\frac{dt}{d\tau}\right)^2 = 0 \qquad (68)$$

or, because Γ^μ_{00} is time independent and $dt/d\tau$ is a constant,

$$\frac{du^\mu}{dt} = -\Gamma^\mu_{00} \qquad (69)$$

It remains to calculate Γ from g. This is tricky (cf. Weinberg eq. (3.4.1)), but finally we get

$$\Gamma^\mu_{00} = \frac{\partial \Phi}{\partial x_\mu}$$ for correspondence with Newtonian case (70)

if we put

$$g_{00} = -(1+2\Phi) \qquad (71)$$

"Post-Newtonian"

The Poisson equation for the potential Φ then takes the form

$$\Delta g_{00} = -8\pi G \varrho \qquad (72)$$

where ϱ is the mass density. Now we can guess the right equation by using relativistic intuition:

(1) Instead of mass, use mass-energy
(2) Because energy, include momentum
(3) Because $g_{00} \rightsquigarrow$ generalize $g_{\mu\nu}$, require 4x4 form

(linear combination of $g_{\mu\nu}$, $\partial g_{\mu\nu}/\partial x^\alpha$, $\partial^2 g_{\mu\nu}/\partial x^\alpha \partial x^\beta$)

=

(tensor form of energy and momentum)

$$G_{\mu\nu} = -8\pi G \, T_{\mu\nu} \qquad (73)$$

Einstein equation

This can be seen in a different way by using the action, S: one requires S from $g_{\mu\nu}$, so one should have $S \propto G^{\mu\nu} g_{\mu\nu}$; identify $G^{\mu\nu} \leftrightarrow T^{\mu\nu}$.

Maximally symmetric time-independent interval

The distance recipe is maximally symmetric when (1) the $g_{\mu\nu}$ is diagonal, (2) its spatial part is spherical; (3) it is time independent, but proof that (3) can be done is omitted. Thus,

$$d^2\tau = g_{00}\, d^2t - g_{rr}\, d^2r - r^2\left(d^2\vartheta + \sin^2\vartheta\, d^2\varphi\right) \qquad (74)$$

A specific choice of coordinates gives

$$g_{00} = \left(1 - 2\frac{M}{r}\right) \quad , \quad g_{rr} = \left(1 - 2\frac{M}{r}\right)^{-1} \; ; \; 2M \equiv R_S \qquad (75)$$

which are Schwarzschild coordinates; but others are possible, e.g. Kruskal coordinates.

Example: motion of radial photons, $d\varphi = d\vartheta = 0$:

$$0 = d^2\tau = \left(1 - 2\frac{M}{r}\right) d^2t - \left(1 - 2\frac{M}{r}\right)^{-1} d^2r \qquad (76)$$

$$\downarrow$$

$$\frac{dr}{dt} = \pm\left(1 - 2\frac{M}{r}\right) \rightsquigarrow \frac{r}{r-2M}\, dr = \pm dt \qquad (77)$$

Solutions: $\frac{r}{r-2M}\, dr = \left(\frac{r-2M}{r-2M} + \frac{2M}{r-2M}\right) dr = dr + 2M\, d\log(r-2M)$

$$t = K \pm \left(r + 2M \log(r-2M)\right)$$

or, rescaling $r \rightsquigarrow r/2M$; $t \rightsquigarrow t/2M$

$$t = K \pm \left(r + \log(r-1)\right) \qquad (78)$$

The zero point of time is unimportant and we get

$$t = r + \log(r-1) \qquad \left(\begin{array}{l}\text{radial outward:}\\ t = r \text{ at } r=\infty\end{array}\right) \qquad (79)$$

<u>Nonradial orbits</u>: just like in the Keplerian case, the orbits are planar because of spherical symmetry; choose $\vartheta = \pi/2$, so that

$$d^2\tau = (1-2\tfrac{M}{r})\,d^2t - (1-2\tfrac{M}{r})^{-1}d^2r - r^2 d^2\varphi \qquad (80)$$

The time independence of $g_{\mu\nu}$ and their φ-independence yields <u>two</u> conserved quantities. Suppose that we increase dt to $dt+\varepsilon$; the consequent change of $d\tau$ is δ, but dr and $d\varphi$ are kept constant; also the $g_{\mu\nu}$ do not change. Then

$$(d\tau+\delta)^2 = g_{00}\,(dt+\varepsilon)^2 - g_{rr}\,d^2r - g_{\varphi\varphi}\,d^2\varphi \qquad (81)$$

$$\Downarrow$$

$$d^2\tau + 2\delta\,d\tau = g_{00}\,d^2t + 2g_{00}\varepsilon\,dt - g_{rr}\,d^2r - g_{\varphi\varphi}\,d^2\varphi \qquad (82)$$

The zero-order term is identically fulfilled therefore

$$2\delta\,d\tau = 2g_{00}\,\varepsilon\,dt \qquad (83)$$

$$\Downarrow$$

$$d\tau \propto g_{00}\,dt \qquad \text{or, for the momentum,} \qquad (84)$$

$$E = \left(1-2\tfrac{M}{r}\right)p_0 \qquad \text{energy conservation!} \qquad (85)$$

Similarly, $d\varphi \rightsquigarrow d\varphi + \varepsilon$ gives

$$\tilde{J} = -g_{\varphi\varphi}\,\frac{d\varphi}{d\tau} = r^2\frac{d\varphi}{d\tau} \quad \text{angular momentum!} \tag{86}$$

$$\tilde{E} = \frac{E}{m} = \left(1-2\frac{M}{r}\right)\frac{dt}{d\tau} \tag{87}$$

(Notes: * derive a redshift from this!)
* What does a conservation law look like in non-diagonal $g_{\mu\nu}$?

Using the above interval $d^2\tau$, we get

$$1 = \left(1-2\frac{M}{r}\right)\left(\frac{dt}{d\tau}\right)^2 - \left(1-2\frac{M}{r}\right)^{-1}\left(\frac{dr}{d\tau}\right)^2 - r^2\left(\frac{d\varphi}{d\tau}\right)^2 \tag{88}$$

$$1 = \tilde{E}^2\left(1-2\frac{M}{r}\right)^{-1} - \left(1-2\frac{M}{r}\right)^{-1}\left(\frac{dr}{d\tau}\right)^2 - r^2\left(\frac{\tilde{J}}{r^2}\right)^2 \tag{89}$$

note: zero for photons!

$$\frac{dr}{d\tau} = \pm\left\{\tilde{E}^2 - \left(1-2\frac{M}{r}\right)\left(1+\frac{\tilde{J}^2}{r^2}\right)\right\}^{1/2} \tag{90}$$

$$\frac{d\varphi}{d\tau} = \tilde{J}/r^2 \tag{91}$$

Division gives the explicit orbit $r(\varphi)$:

$$\frac{dr}{d\varphi} = \pm\frac{r^2}{\tilde{J}}\left\{\tilde{E}^2 - \left(1-2\frac{M}{r}\right)\left(1+\tilde{J}^2/r^2\right)\right\}^{1/2} \tag{92}$$

By comparison with the Newtonian case $(2M/r) \rightsquigarrow 0$ we see that

$$\tilde{E}^2/\tilde{J}^2 = 1/b^2 \quad ; \quad b = \text{impact parameter} \tag{93}$$

$$\left(r^{-2}\frac{dr}{d\varphi}\right)^2 + r^{-2}\left(1-2\frac{M}{r}\right) + \left(1-2\frac{M}{r}\right)\tilde{y}^{-2} = b^{-2} \qquad (94)$$

For photons, the third term disappears; $\tilde{y} \equiv y/m$ and so $(1/\tilde{y}) \to 0$, wherefore

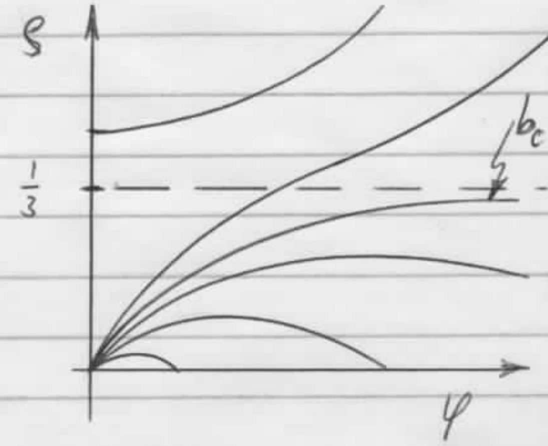
$$\left(\frac{1}{r^2}\frac{dr}{d\varphi}\right)^2 + \frac{1}{r^2}\left(1-2\frac{M}{r}\right) = \frac{1}{b^2} \qquad (95)$$

Substitution of $\varrho \equiv 2M/r$ gives

$$\left(\frac{d\varrho}{d\varphi}\right)^2 + \varrho^2(1-\varrho) = \varrho_\infty^2 \qquad \varrho_\infty \equiv \frac{2M}{b} \qquad (96)$$

The character of the solutions can be found by studying the phase plane: $\frac{d\varrho}{d\varphi}$ versus ϱ.
There is a singular point at $\varrho = 2/3$, which corresponds to a minimum pericentre distance $r_c = 3M$; any photon inside that radius is caught! The corresponding critical impact parameter is $\frac{d\varrho}{d\varphi} = 0$, $\varrho = 2/3$:

$$\varrho_c^2 = \frac{4}{9}\left(1-\frac{2}{3}\right) = \frac{4}{27} \rightsquigarrow b_c = \frac{3}{2}\sqrt{3} = 5.196152 \times \frac{1}{2} \qquad (97)$$



NB $\varrho \equiv 2M/r$

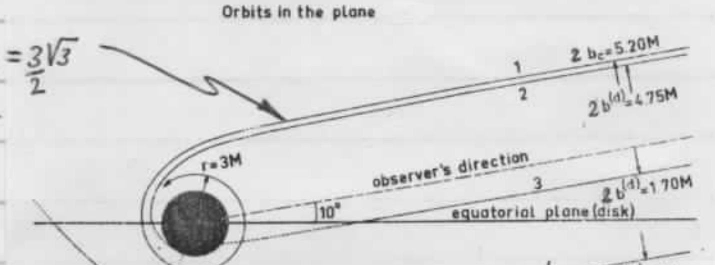
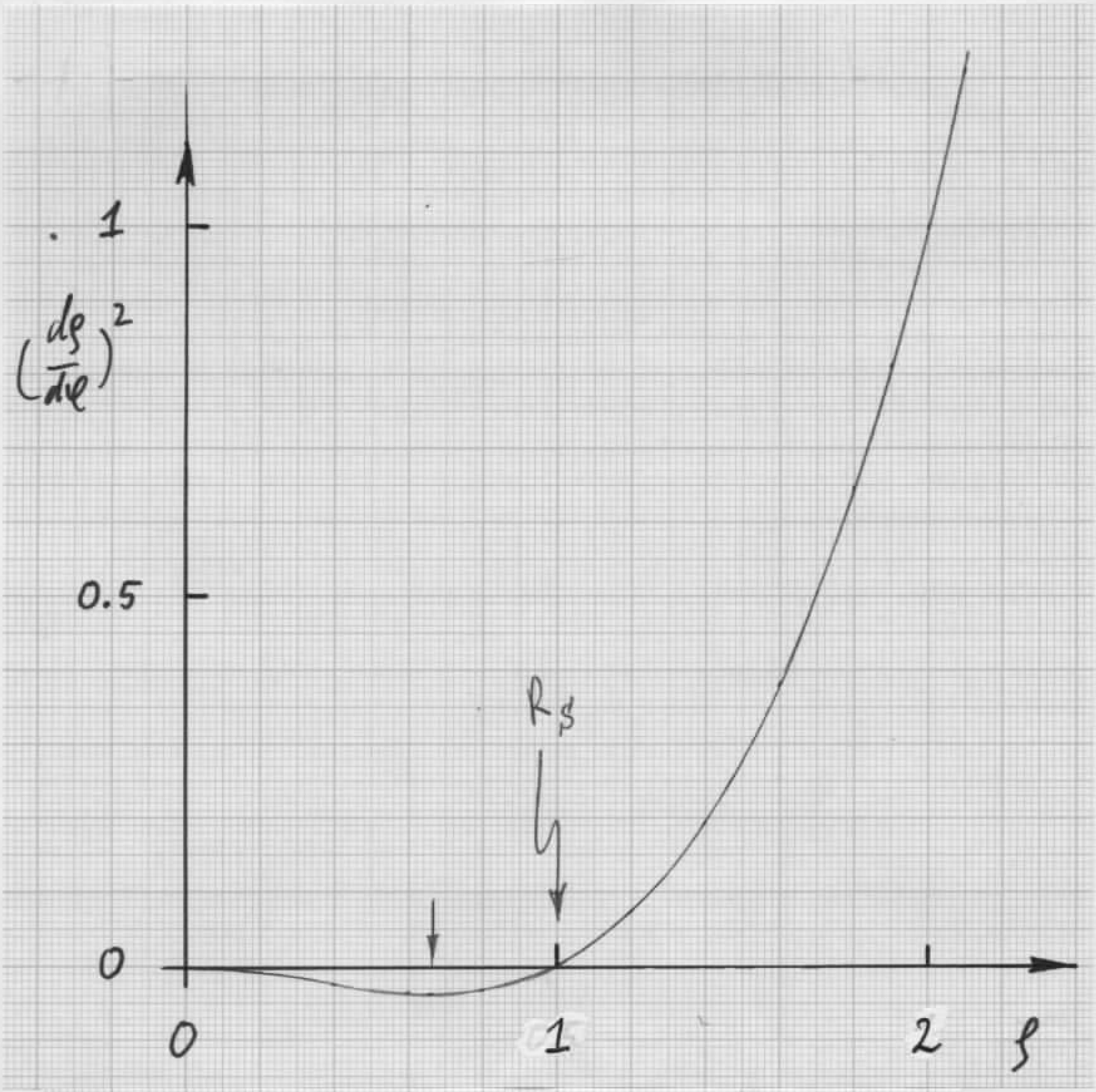
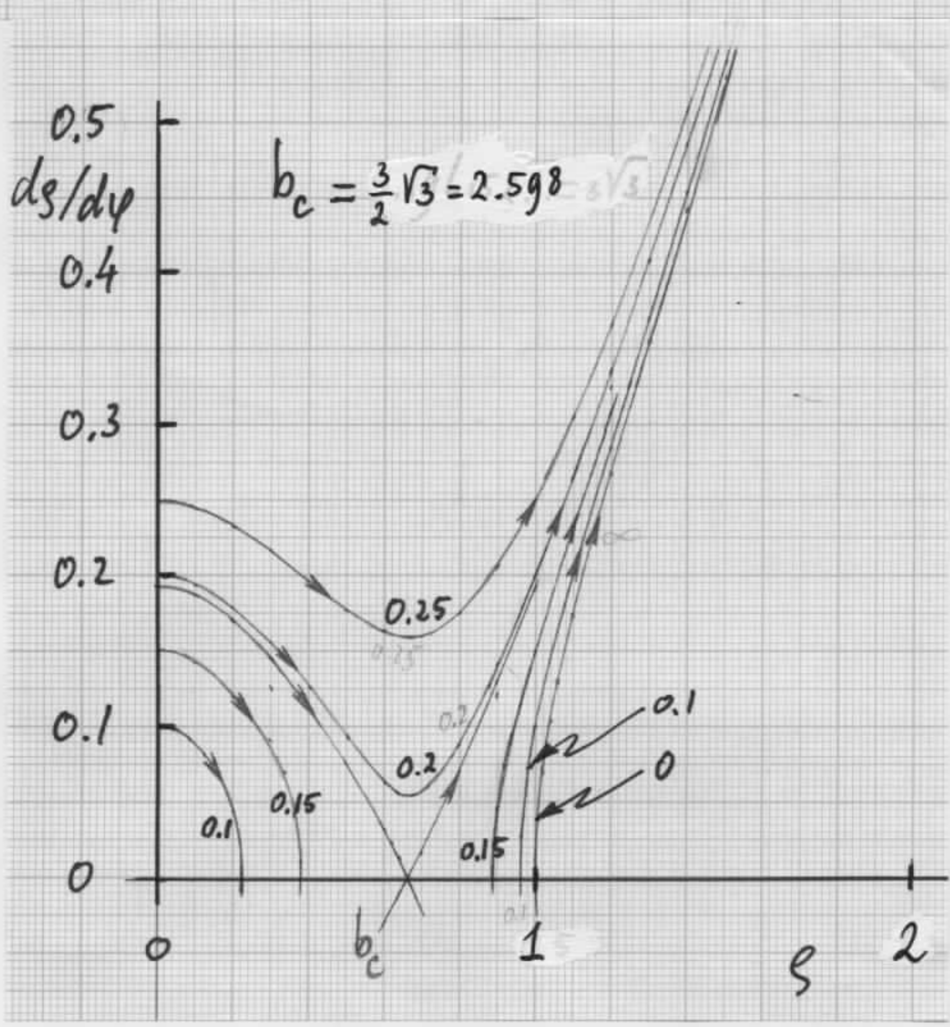


Fig. 4. Illustrative orbits in the plane Trajectory 1 has the critical impact parameter and circles infinitely around the black hole; trajectories 2 and 3 give direct images, trajectory 4 gives a secundary image


1
0.5
0
$\left(\frac{d\rho}{d\varphi}\right)^2$
R_S
0
1
2
ρ


$b_c = \frac{3}{2}\sqrt{3} = 2.598$
0.5
$d\rho/d\varphi$
0.4
0.3
0.2
0.1
0
0.25
0.2
0.15
0.1
0.15
0.1
0
b_c
0
1
2
ρ

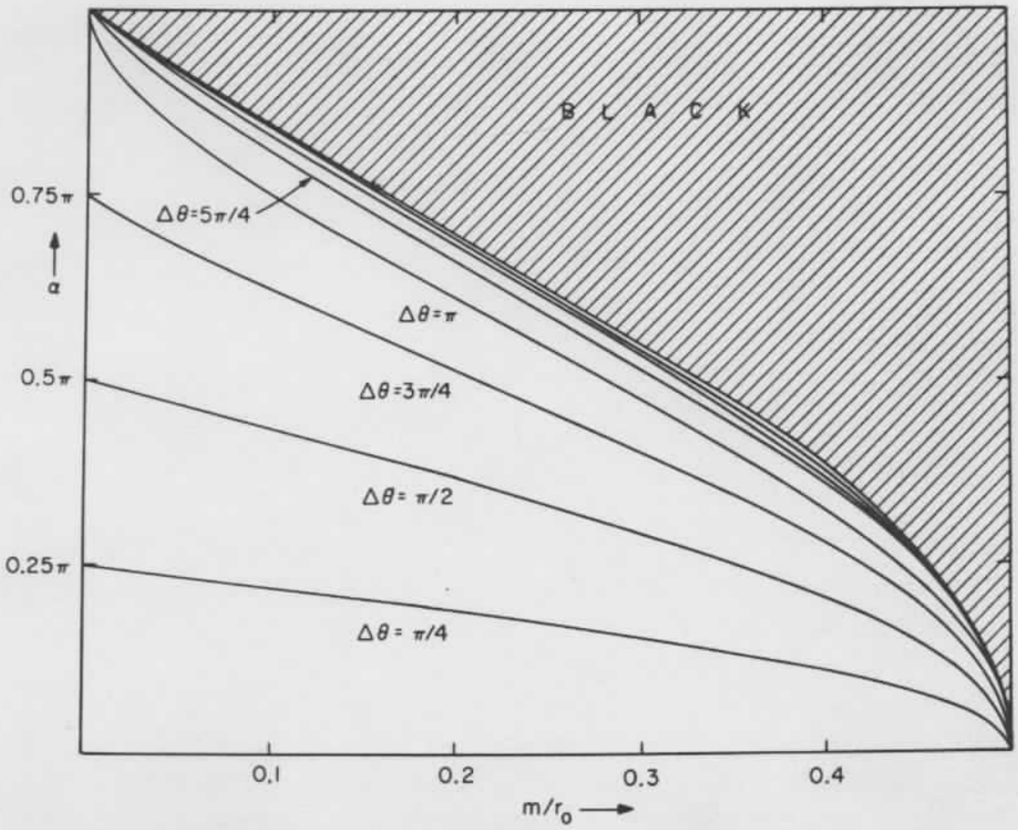


FIG. 1. The optical appearance of the external universe to a static observer at radius r_0. α is the observed angle of a photon relative to the outward radial direction. $\Delta\theta$ is the change in polar angle along a photon trajectory, hence it is the polar angle of the emission of the photon. Curves of $\Delta\theta = n\pi/4$ pile up along the curve $\alpha = \alpha_{max}$, corresponding to the angle of photons from the unstable circular photon orbit at $r=3m$. The sky is black between α_{max} and the inward direction ($\alpha = \pi$).

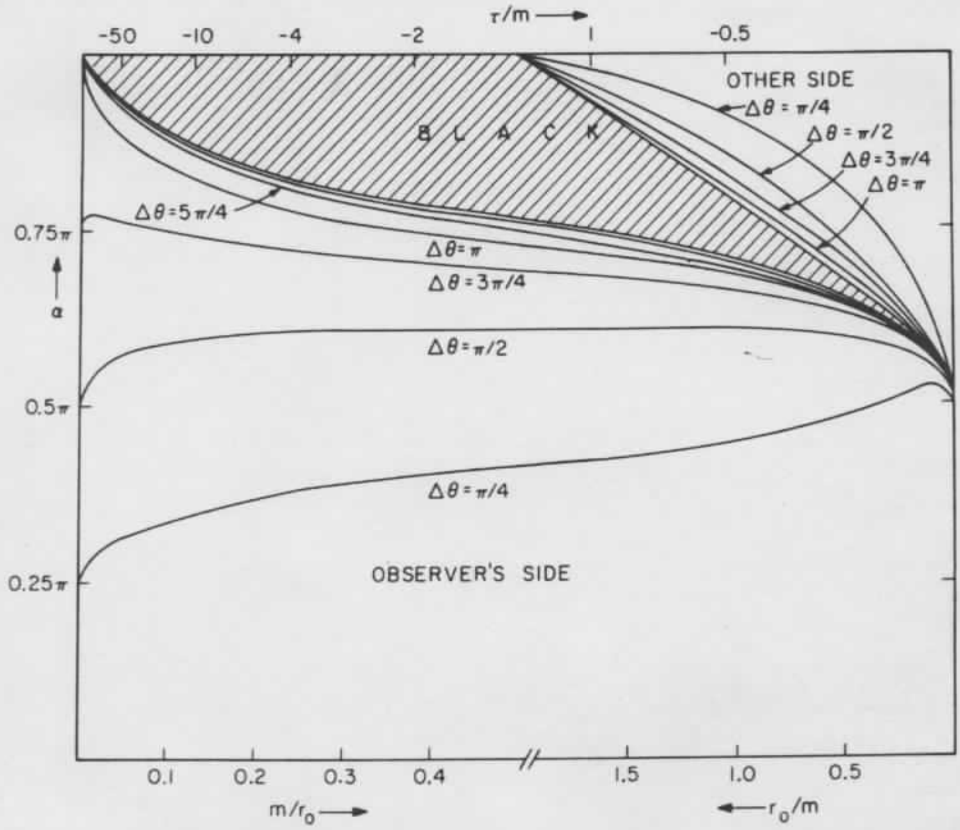


FIG. 3. The appearances of the two "universes" seen by the free-fall observer. Details as in Fig. 1. The universe seen at large r_0 is the home of the observer. That first seen upon crossing the horizon is represented by the "other side" of the Kruskal diagram.

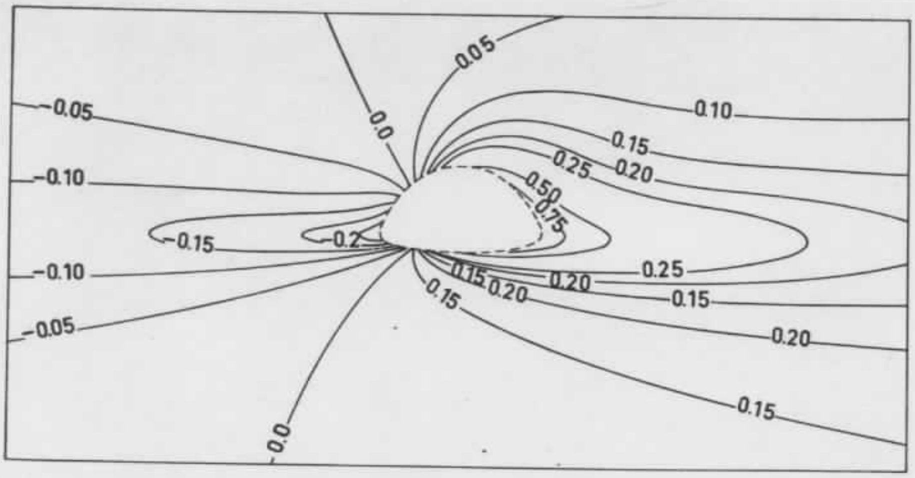


Fig. 8. Curves $\{z = \text{constant}\}$ as seen by an observer at 10° above the disk's plane

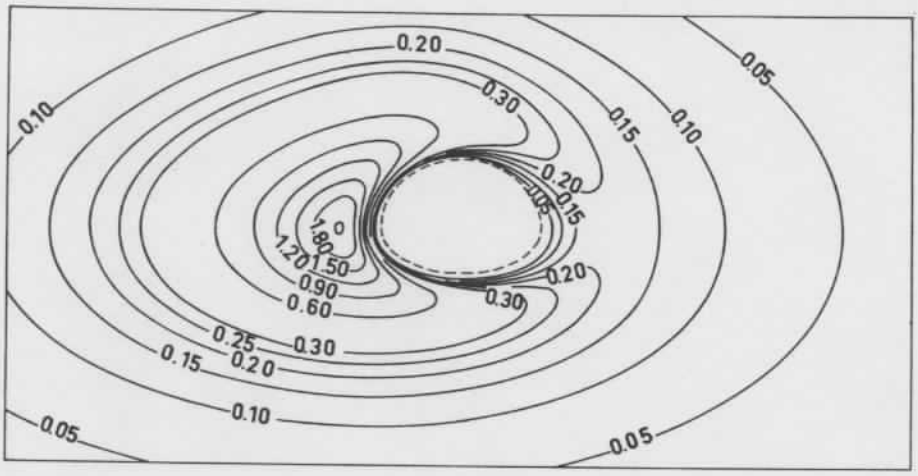


Fig. 9. Curves of constant flux in units of $F_{S_{max}}$, as senn by an observer at 30° above the disk's plane. Dashed line: the apparent inner edge of the disk (flux = 0)

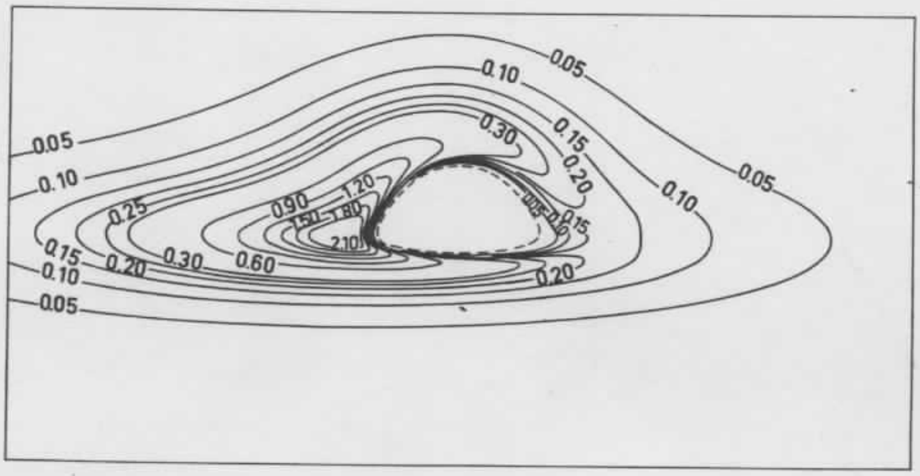


Fig. 10. Curves of constant flux in units of $F_{S_{max}}$, as seen by an observer at 10° above the disk's plane. The maximum value within the area limited by $F_0 = 2.10$ is 2.62

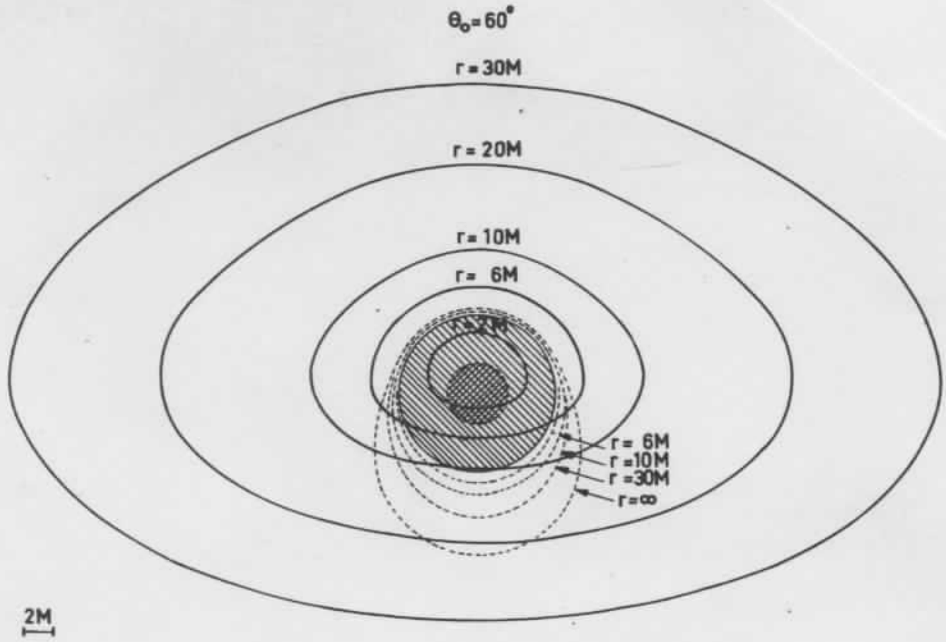


Fig. 5. Isoradial curves representing rays emitted at constant radius from the hole, as seen by an observer at 30° above the disk's plane. Full lines: direct images; dashed lines: secundary images

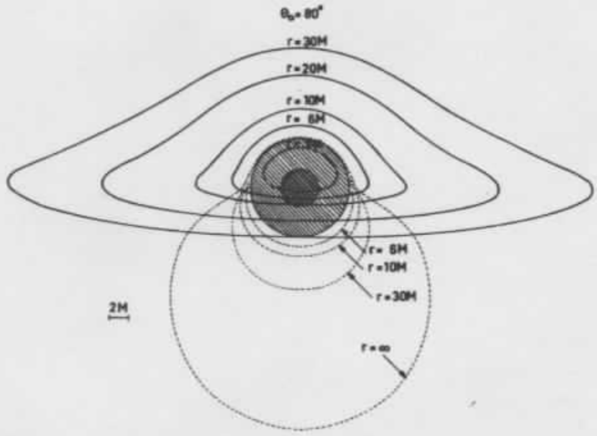


Fig. 6. Isoradial curves as seen by an observer at 10° above the disk's plane

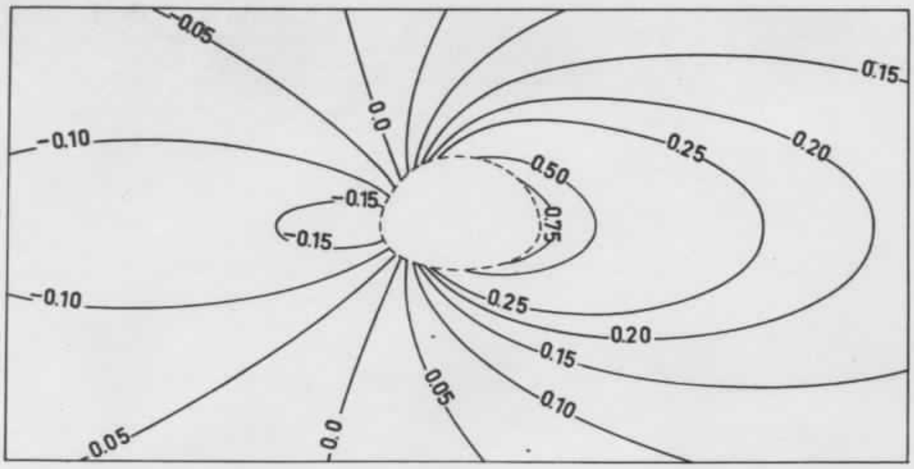


Fig. 7. Curves $\{z = \text{constant}\}$ as seen by an observer at 30° above the disk's plane. Dashed line: the apparent inner edge of the disk

Exercise: By comparison with the equations for the classical Kepler problem (e.g. Landau & Lifschitz I, § 15) write down the equations of motion (90, 91) for a photon near a black hole in terms of an effective potential Φ_{BH}.

Exercise: Derive the equivalent of eq. (96) for a massive particle.

Exercise: What is Φ_{BH} for a massive particle? What causes the "pit in the potential"?

The role of the density

From density, via Lorentz trf. properties, to the stress-energy tensor;
analogy with Newtonian case → Einstein eq.

We note that we get one Lorentz factor γ for each vector index; therefore, if an object transforms such that it gets

γ^0	it must be a scalar		1
γ^1	- - - - vector		p^μ
γ^2	- - - - tensor	etc.	$S^{\mu\nu}$

For the mass density ρ we immediately see that

$$\rho' = \gamma^2 \rho \qquad (26)$$

(or, strictly, $\rho' = \gamma^2 \rho + \dots$) because we get one Lorentz factor from the Lorentz-Fitzgerald contraction of the volume occupied by the mass, and another γ from the transformation of the mass: $m' = \gamma m$.
Thus, ρ must be the (00) component of a tensor.

The analogue of the (0) component of a vector, is the main diagonal of a tensor. So we build

$$\left(\begin{array}{c|ccc} \rho & 0 & 0 & 0 \\ \hline 0 & & & \\ 0 & & \phi & \\ 0 & & & \end{array}\right) \rightsquigarrow \begin{pmatrix} \rho & 0 & 0 & 0 \\ 0 & P & & \phi \\ 0 & & P & \\ 0 & \phi & & P \end{pmatrix} \rightsquigarrow \begin{pmatrix} \rho+P & \phi \\ \phi & \phi \end{pmatrix} + P\begin{pmatrix} -1 & & & \phi \\ & 1 & & \\ & & 1 & \\ \phi & & & 1 \end{pmatrix} \qquad (27)$$

The P's come in as velocity moments, averages over the particle distribution:

<u>sketch</u> of tensor

here we expect $\int f \vec{v}\, d^3p$

here we expect $\int f\, v_i v_j\, d^3p$

here we put $\int f\, d^3p = \rho$

The exact construction can be found by starting with a diagonal tensor (= "rest system"):

$$\mathcal{T} \equiv \begin{pmatrix} \rho & & & \phi \\ & p & & \\ & & p & \\ \phi & & & p \end{pmatrix} \qquad (28)$$

Lorentz transformation with L. trf. $\mathcal{L}$ gives according to classical matrix algebra

$$\mathcal{T}' = \mathcal{L}\,\mathcal{T}\,\mathcal{L}^T \qquad (29)$$

$$\mathcal{T}' = \begin{pmatrix} \gamma & -\beta\gamma & & \phi \\ -\beta\gamma & \gamma & & \\ & & 1 & \\ \phi & & & 1 \end{pmatrix} \begin{pmatrix} \rho & & & \phi \\ & p & & \\ & & p & \\ \phi & & & p \end{pmatrix} \begin{pmatrix} \gamma & -\beta\gamma & & \phi \\ -\beta\gamma & \gamma & & \\ & & 1 & \\ \phi & & & 1 \end{pmatrix}$$

$$= \begin{pmatrix} \gamma^2(\rho+\beta^2 p) & -\beta\gamma^2(\rho+p) & & \phi \\ -\beta\gamma^2(\rho+p) & \gamma^2(p+\beta^2\rho) & & \\ & & p & \\ & \phi & & p \end{pmatrix} \qquad (30)$$

This leads to the supposition that

$$T^{\alpha\beta} = p\,\eta^{\alpha\beta} + (p+\rho)\,u^\alpha u^\beta \qquad (31)$$

which is the requisite tensor generated by ρ and the Lorentz symmetry.

Note the course of the argument:

1) Many-particle system $\rightarrow$ phase space
2) Taking averages in phase space is OK only if we Lorentz transform the momenta
3) Lorentz trf. of $\rho = \int f d^3p$ gives γ^2
4) Therefore ρ is part of a tensor $\overline{T}$
5) In the rest system, $\overline{T}$ is diagonal (analogy with (0) component of vector: main diagonal)
6) Diagonal elements are $\int f v_i v_j d^3p = P$ pressure
7) Explicit calculation of $\overline{T}$, by means of Lorentz transformation of the diagonal form, gives $T^{\alpha\beta}$.

What role does $T^{\alpha\beta}$ play in the equations of motion? In the classical case, we have Poisson's Equation which connects mass density with gravitational potential:

$$\Delta \Phi = 4\pi G\rho \qquad (32)$$

Thus, we expect that the general case is

$$(\text{second deriv's of } g^{\alpha\beta}) = T^{\alpha\beta} \qquad (33)$$

which is the basic form of the Einstein equation.
Note that this does not answer the question why matter and gravity couple in this way!
This is at present an unsolved problem.
We need quantum gravity to solve this (probably).