150191

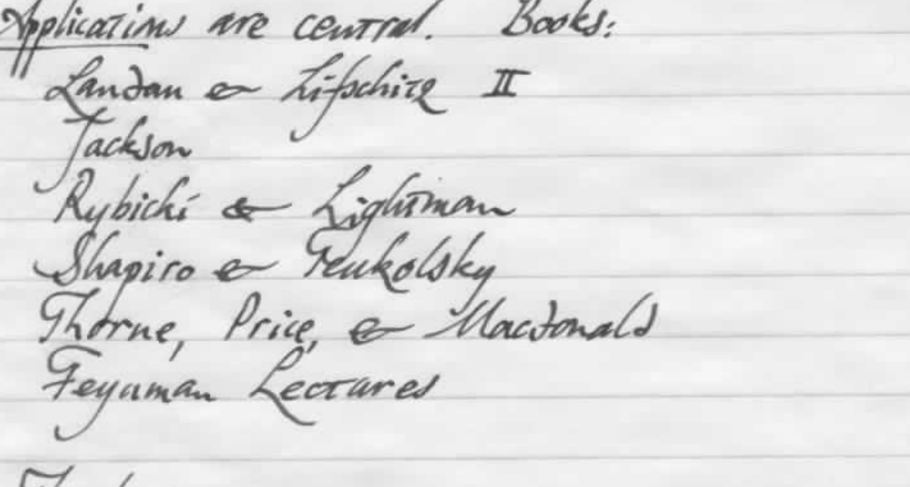
Application of Relativity in Astrophysics

I. Introduction to Special Relativity

1. Applications are central. Books:
   - Landau & Lifschitz II
   - Jackson
   - Rybicki & Lightman
   - Shapiro & Teukolsky
   - Thorne, Price, & Macdonald
   - Feynman Lectures

2. The basics

Classical mechanics: $\bar{x}$ & t, Galilei invariance

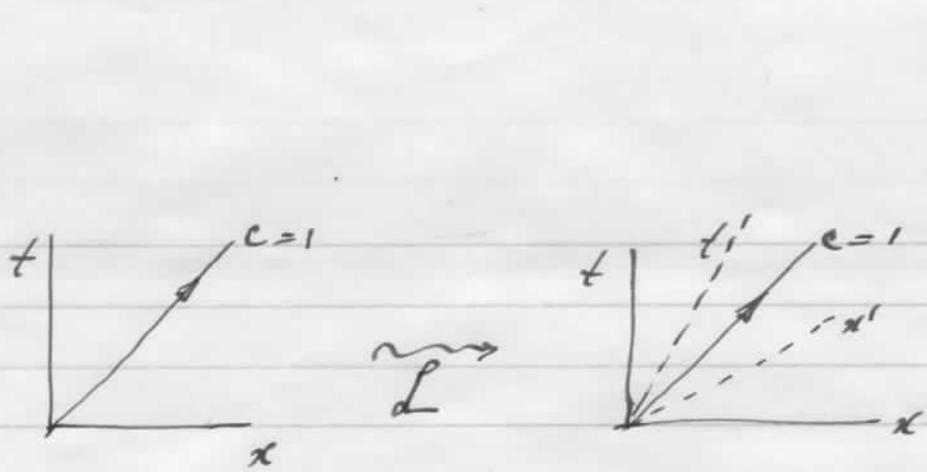


Fact: Michelson-Morley 1888

c is invariant

Choose $c \equiv 1$, then light rays are $45°$ diagonals in all xt-diagrams (light cone). Therefore, $x = \pm ct$ must always bisect the xt-axes. $\rightarrow$ no Galilei invariance:

150191



To keep $x=ct$ a bisectrix, t must change too!

Because $[c]=[m/s]$, space & time must mix if you want to keep a velocity invariant.

The transformation $\mathcal{L}$ is like a rotation, except t and x rotate in opposite directions:

$$t' = \gamma(t-\beta x) \qquad x' = \gamma(x-\beta t) \qquad (1)$$

What are β and γ? $(t',x') = \mathcal{L}(t,x)$, so the norm of $\mathcal{L}$ must be unity:

$$\begin{vmatrix} \gamma & -\beta\gamma \\ -\beta\gamma & \gamma \end{vmatrix} = \gamma^2 - \beta^2\gamma^2 = 1 \rightarrow \gamma^2 = \frac{1}{1-\beta^2} \qquad (2)$$

Lorentz trf

Correspondence with Galilei: small values of β give $\gamma^2 = 1$ and

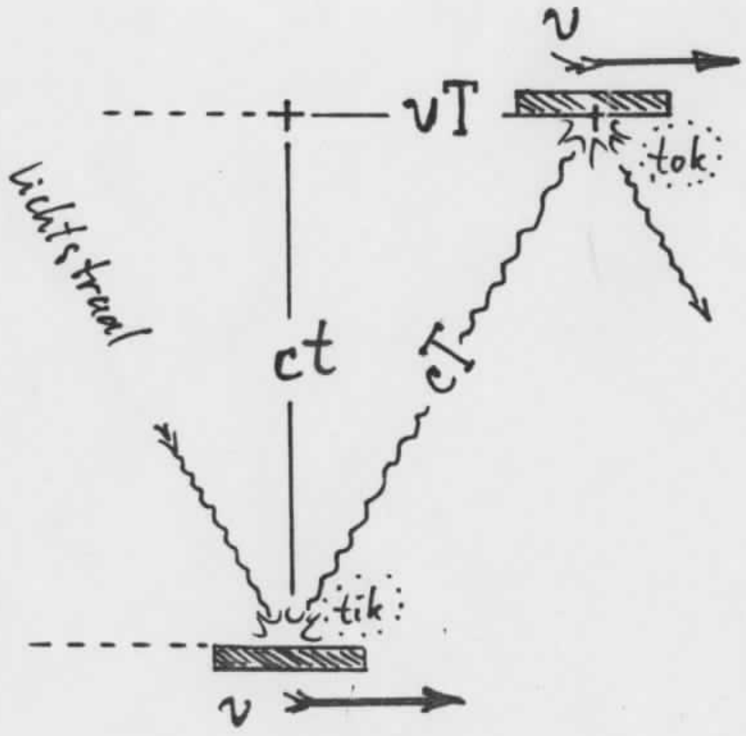
$$x' \simeq x - \beta t \qquad (3)$$

Therefore, β = relative velocity of (t',x') and (t,x)

$$\beta = v/c \qquad (4)$$

"Rotation" in (1): $\gamma \equiv \cosh \zeta$; $\gamma\beta \equiv \sinh \zeta$; $\beta = \tanh \zeta$

ζ = rapidity



$$c^2T^2 = c^2t^2 + v^2T^2$$

$$(c^2 - v^2)T^2 = c^2t^2$$

$$T^2 = \frac{c^2t^2}{c^2 - v^2}$$

$$T = \frac{t}{\sqrt{1 - v^2/c^2}}$$

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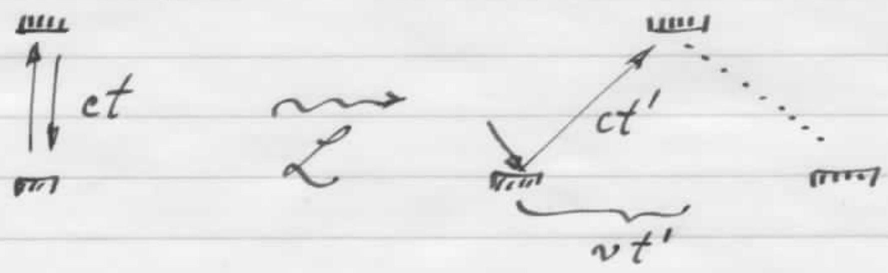
Mixing of space & time: the Lorentz clock.



$$c^2t'^2 = c^2t^2 + v^2t'^2 \rightsquigarrow t'^2 = t^2 + \beta^2 t'^2$$

$$t' = \gamma t \qquad (4^a)$$

<u>time dilatation</u>

<u>Exercise</u>: Lorentz-Fitzgerald contraction

$$x' = x/\gamma \qquad (4^b)$$

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If L is a pseudo-rotation, it has an invariant. This is given by

$$t^2 - x^2 = \text{constant} \qquad (5)$$

Lorentz invariant. Verify explicitly. Space and time mix.

Notice peculiar inner product:

$$\underset{0\cdots3}{x^\mu x_\mu} = x^0 x_0 - \underset{1\cdots3}{x^i x_i} = t^2 - x^2 - y^2 - z^2 \qquad (6)$$

$+ - - -$ convention

Note: only one opposite sign; one time coordinate, else subspaces like

$$\underbrace{t^2 + \lambda^2}_{\text{circle}} - x^2 - y^2 \qquad (7)$$

circle: closed timelike curves; causality!

There is only one time.

*

‖ Everything comes in fours, and inner product $+---$. Mixing of 0-component with i-components. ‖

This is what makes relativity counter-intuitive!

For example, circles:

$$x^\mu x_\mu = \text{constant} \quad \text{(definition!)} \qquad (8)$$

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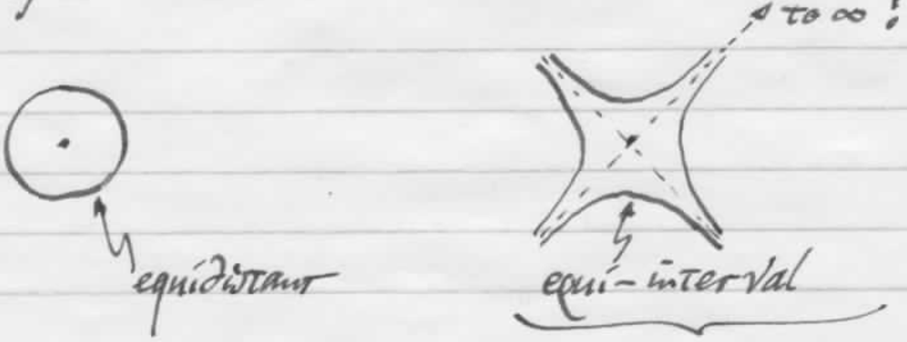
"Distance" $\sqrt{x^\mu x_\mu}$ is called <u>interval</u>.

Pythagoras $x^2+y^2=r^2$

Minkowski $t^2-x^2=s^2$



light "cone" { This is the shape of a <u>local patch</u>
Etch this in your brain!

Everything comes in <u>fours</u>: $4^n =$ 1 scalar
4 vector (or 2-spinor)
16 tensor

Interval is a Lorentz scalar. Coordinates are a Lorentz vector (t, x, y, z). Electromagnetic field is a Lorentz tensor but vector potential is L-vector. or 4-vector

<u>Phase</u> is a Lorentz scalar (counting only):

$$e^{-i(\vec{k}\cdot\vec{x}-\omega t)} = e^{-ik^\mu x_\mu} \tag{9}$$

Inner product again! Suggests 4-vector $(\omega, \vec{k})$.

$$\omega^2 - k^2 = \text{const} \tag{10}$$

dispersion relation
+ De Broglie: Klein-Gordon & Dirac eqs

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Suggests from classical e.m. for light:

$$\omega^2 = k^2 \rightarrow \omega = k \qquad (11)$$

"massless"

Likewise, three-momentum cannot be Lorentz invariant but must be extended:

$$\vec{p} = m\vec{v} \qquad \text{classical} \quad (\text{NB: } c \equiv 1!) \qquad (12)$$

Test a 4-vector $(\psi, \vec{p})$ under Lorentz transformation. If initially no momentum, then $(\psi, 0)$; now transform to a small velocity β. Then

$$\psi' = \gamma\psi \; ; \quad p' = -\beta\gamma\psi \simeq -\beta\psi \qquad (13)$$

At small β, the momentum must be $-m_0\beta$, so that $\psi_0 \simeq m_0$. Lorentz invariance requires

$$\psi_0^2 - p_0^2 \simeq m_0^2 \rightarrow \psi_0 \simeq \sqrt{m_0^2 + p_0^2} \qquad (14)$$

For small v, we get $p_0 \simeq m_0 v$ and

$$\psi_0 \simeq m_0\sqrt{1+v^2} \simeq m_0 + \tfrac{1}{2} m_0 v^2 \qquad (15)$$

The fourth component $(\equiv p^0)$ is <u>energy</u>

$$p^\mu = (E, \vec{p}) \qquad (16)$$

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Lorentz invariance:

$$E^2 - p^2 = \text{const.} \equiv m_0^2 \qquad (17)$$

Momentum and energy mix, just like space and time mix!
The interval is invariant $\leftrightarrow$ the rest mass is invariant
The origin of mass: depends on how m_0 propagates $\Rightarrow$ the propagation determines m!
cf. origin of m in high-energy physics

$$\vec{p} = \gamma m_0 \vec{\beta} \; ; \quad E = \gamma m_0 \qquad (18)$$

(or $E = \gamma m_0 c^2$ if $c = 1$ units)

$E = m_0$ if $\beta = 1$ $(= mc^2)$

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A bit more systematic kinematics, now! Start with the 4-coordinate vector $(t, \vec{x})$. By analogy with the 3-velocity $\vec{v} \equiv d\vec{x}/dt$ we should like to get a velocity. But dt is not a Lorentz invariant! Only $t^2 - x^2$ is. Or, in infinitesimal form,

$$dt^2 - dx^2 = ds^2 \qquad (19)$$

notation: $dt^2 \neq 2t\,dt$ &c. ! but $(dt)^2$

Now $dx/dt \equiv v$, so that

$$ds^2 = (1 - v^2)\,dt^2 \equiv \frac{1}{\gamma_v^2}\,dt^2 \; ; \quad \gamma_v \equiv (1-v^2)^{-1/2} \qquad (20)$$

Therefore we define the (invariant!) proper time by

150191

$$d\tau \equiv \frac{1}{\gamma_v}\,dt \qquad (21)$$

Then we can define the <u>four-velocity</u> u^μ as

$$u^\mu \equiv \frac{dx^\mu}{d\tau} \Rightarrow \qquad (22)$$

$$u^0 = \frac{dt}{d\tau} = \gamma_v \qquad (23)$$

$$u^i = \frac{dx^i}{d\tau} = \frac{dt}{d\tau}\,\frac{dx^i}{dt} = \gamma_v\,\vec{v} \qquad (24)$$

<u>Prove</u> that 4-accel. $\perp$ 4-veloc.
L & L II § 7

$\ast$

To get to 3-velocities: Jackson § 11.4

$\ast$

Lorentz trf. without rotation $\equiv$ <u>Lorentz boost</u>

$$x' = \gamma(x-\beta t) \quad ; \quad t' = \gamma(t-\beta x) \qquad (25)$$

$$v' = \frac{dx'}{dt'} = \frac{dx-\beta dt}{dt-\beta dx} = \frac{v-\beta}{1-v\beta} \qquad (26)$$

Velocity addition in <u>one</u> coordinate direction

$$\text{"}u+v\text{"} = \frac{u+v}{1+uv}$$

max velocity = 1 ! <u>proof</u>, not experiment
<u>retardation</u> → <u>radiation</u> (27)

Now acceleration! If you already move with speed u and you get a little bump δu, then

$$\text{"}u+\delta u\text{"} = \frac{u+\delta u}{1+u\delta u} \simeq (u+\delta u)(1-u\delta u) \simeq u+\delta u - u^2\delta u$$
$$= u + \frac{1}{\gamma_u^2}\,\delta u \qquad (28)$$

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As seen in an external frame, then, u increases by the amount $\gamma_u^{-2}\,\delta u$. In the rest frame of the moving body, the velocity goes from zero to δu in a time $\delta t'$; this is the acceleration A' as experienced by the body, so that

$$\frac{du}{dt'} = \gamma_u^{-2} A' \qquad (29)$$

or, because $dt' = \gamma_u dt$ at a fixed spatial point x,

$$\frac{du}{dt} = \frac{A'}{\gamma_u^3} = (1-u^2)^{3/2} A' \qquad (30)$$

Exercise: the "twin effect" for A' = one gee!

$$\int (1-u^2)^{-3/2}\,du = u(1-u^2)^{-1/2} = A't \qquad (31)$$

$$u = \frac{A't}{\sqrt{1+A'^2t^2}} = \frac{dx}{dt} \qquad \text{if } u=0 \text{ @ } t=0 \qquad (32)$$

$$x = \sqrt{1+A'^2t^2} - 1 \qquad \text{if } x=0 \text{ @ } t=0 \qquad (33)$$

$$dt' = \gamma_u\,dt\ ;\quad \gamma_u = (1+A'^2t^2)^{1/2} \qquad (34)$$

$$t' = \log\left(A't + \sqrt{1+A'^2t^2}\right) \qquad (35)$$

Is $\lim\limits_{t\to\infty} u = 1$ above a radiating surface?

If $\frac{A'}{m} = 9.5\ ms^{-2}$, then $t=1 \mathrel{\hat=}$ one year, $x=1 \mathrel{\hat=}$ one lightyear.

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Velocity addition in general 3D (Jackson p. 523): if velocity is $\vec{u}'$ in system K' which moves with $\vec{\beta}$ with respect to K, then in K one observes a velocity $\vec{u}$ according to

$$\left.\begin{aligned} dt &= \gamma\,(dt' + \vec{\beta}\cdot d\vec{x}\,') \\ d\vec{x} &= \gamma\,(d\vec{x}\,' + \vec{\beta}\,dt') \end{aligned}\right\} \qquad (36)$$

$$\vec{u} = \frac{d\vec{x}}{dt} = \frac{d\vec{x}\,' + \vec{\beta}\,dt'}{dt' + \vec{\beta}\cdot d\vec{x}\,'} = \frac{\vec{u}\,' + \vec{\beta}}{1+\vec{\beta}\cdot\vec{u}\,'} \qquad (37)$$

Split into components parallel & perpendicular to $\vec{\beta}$, this gives

$$u_\parallel \equiv \vec{u}\cdot\frac{\vec{\beta}}{\beta} = \frac{\vec{u}\,'\cdot\vec{\beta}/\beta + \beta}{1+\vec{\beta}\cdot\vec{u}\,'} = \frac{u'_\parallel + \beta}{1+\vec{\beta}\cdot\vec{u}\,'} \qquad (38)$$

$$\vec{u}_\perp = \vec{u} - u_\parallel\vec{\beta}/\beta = \vec{u} - \left(\vec{u}\cdot\frac{\vec{\beta}}{\beta}\right)\frac{\vec{\beta}}{\beta} = \frac{\vec{u}\,' + \vec{\beta} - u_\parallel\vec{\beta}/\beta - \vec{\beta}}{1+\vec{\beta}\cdot\vec{u}\,'} =$$

$$= \frac{\vec{u}\,' - (\vec{u}\,'\cdot\vec{\beta}/\beta)\,\vec{\beta}/\beta}{\cdots} = \frac{\vec{u}\,'_\perp}{\gamma\,(1+\vec{\beta}\cdot\vec{u}\,')} \qquad (39)$$

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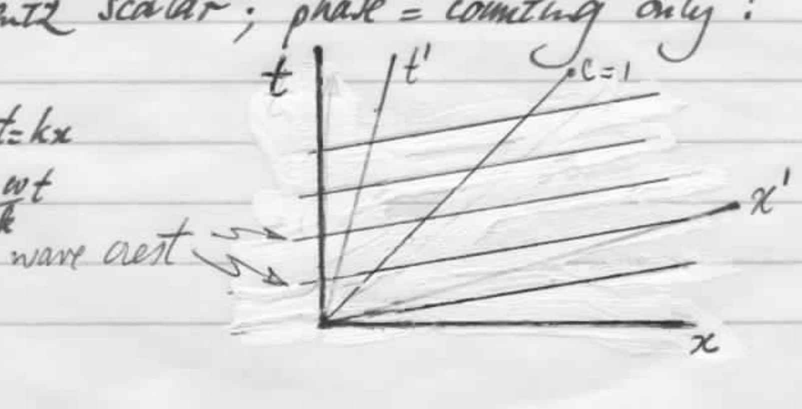
More 4-vectors: back to (9). Wave vector via phase as Lorentz scalar; phase = counting only!

stationary phase $\omega t = kx$
phase velocity $x = \frac{\omega}{k} t$



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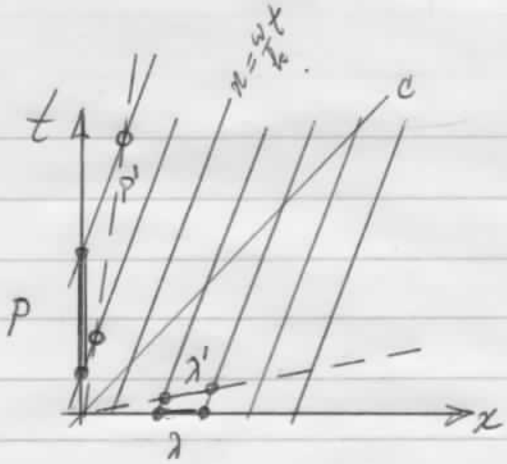


space & time mix

Period & wavelength mix

$(\omega, \vec{k})$ wave 4-vector

$$\begin{aligned} \omega' &= \gamma(\omega - \beta k_x) \\ k_x' &= \gamma(k_x - \beta\omega) \\ k_y' &= k_y \; ; \quad k_z' = k_z \end{aligned} \qquad (40)$$

$$\begin{aligned} k'^2 = k_x'^2 + k_y'^2 + k_z'^2 &= \gamma^2(k_x - \beta\omega)^2 - k_x^2 + k^2 \\ &= \gamma^2\beta^2 k_x^2 - \gamma^2\beta\omega k_x + \gamma^2\beta^2\omega^2 + k^2 \\ &= \gamma^2\beta^2 k_x^2 - \gamma^2\beta\omega k_x + \gamma^2 k^2 \end{aligned}$$

Using the direction of the wave:

$$\cos\vartheta \equiv k_x / k \qquad (41)$$

$k = \omega$ dispersion!

we get

$$\omega' = \gamma(\omega - \beta k \cos\vartheta) = \gamma\omega(1 - \beta\cos\vartheta) \qquad (42)$$

Doppler

If $\vartheta = 0$, classical Doppler

$$\omega' = \omega\,\gamma(1-\beta) = \omega\,(1-\beta^2)^{-1/2}(1-\beta) = \omega\sqrt{\frac{1-\beta}{1+\beta}} \qquad (43)$$

Frequency & direction mix!

15.01.91

$$\cos\vartheta' = k_x'/k' = \frac{\gamma(k_x - \beta\omega)}{\omega'} = \frac{\omega}{\omega'}\gamma(\cos\vartheta - \beta)$$

dispersion

Using (43), this produces

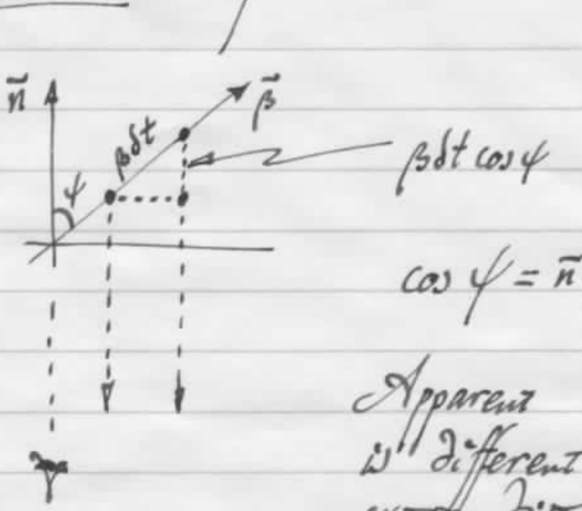
$$\cos\vartheta' = \frac{\cos\vartheta - \beta}{1-\beta\cos\vartheta} \qquad (44)$$

Aberration; check $\vartheta = 0$, $\vartheta = \pi$.

* Appearance neutron star
* rel. beaming

Application: photon surfing '89 A&A 216, 294

Application: superluminal motion



$\cos\psi = \vec{n}\cdot\vec{\beta}/\beta$

Apparent transverse velocity is different, because a certain extra distance must be covered by the light. Instead of taking a time δt, it takes δt plus the light travel distance $\beta\,\delta t\cos\psi$,

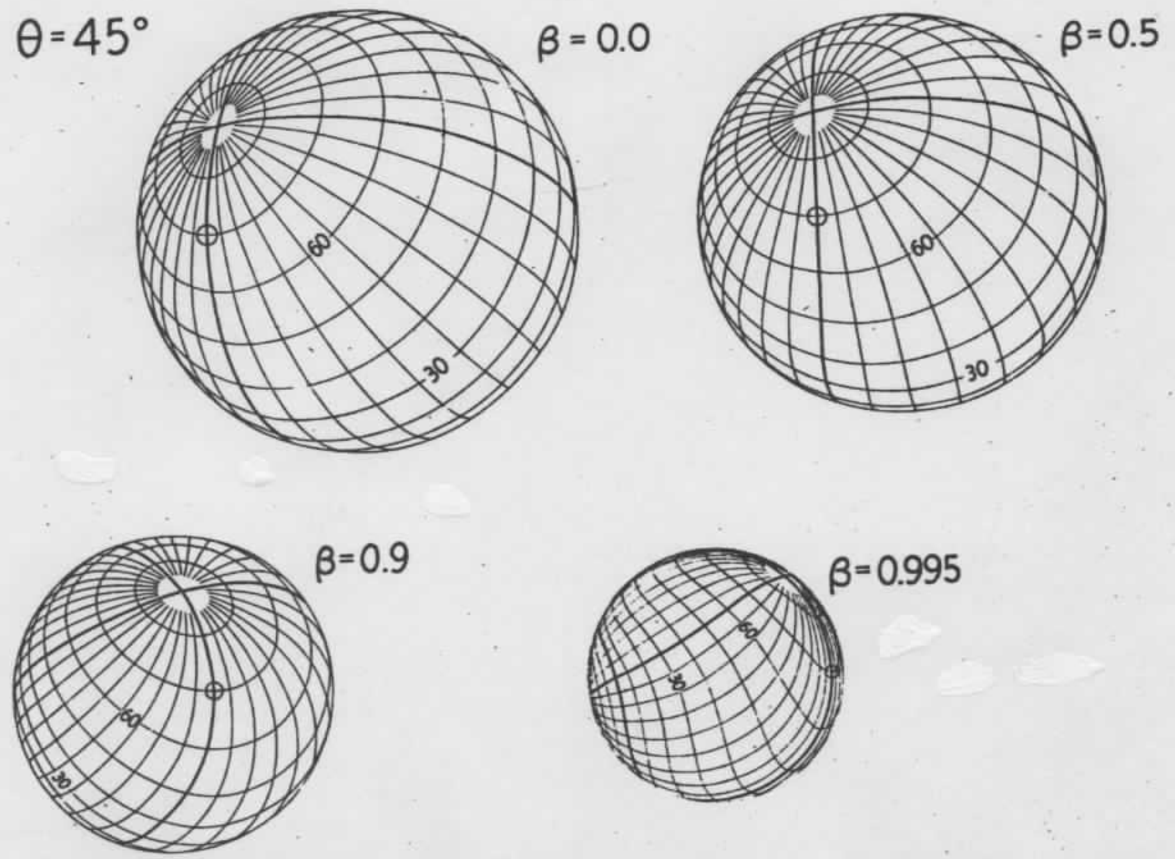


FIG. 4. Appearance of a sphere approaching an observer at various speeds β. The center of the sphere is moving along a line which is at a distance of one sphere diameter from the observer. As described in the text, the sphere is tilted by 70° to display the north pole. The direction from the point circled on the sphere to the observer makes an angle θ equal to 45° to the line of motion.

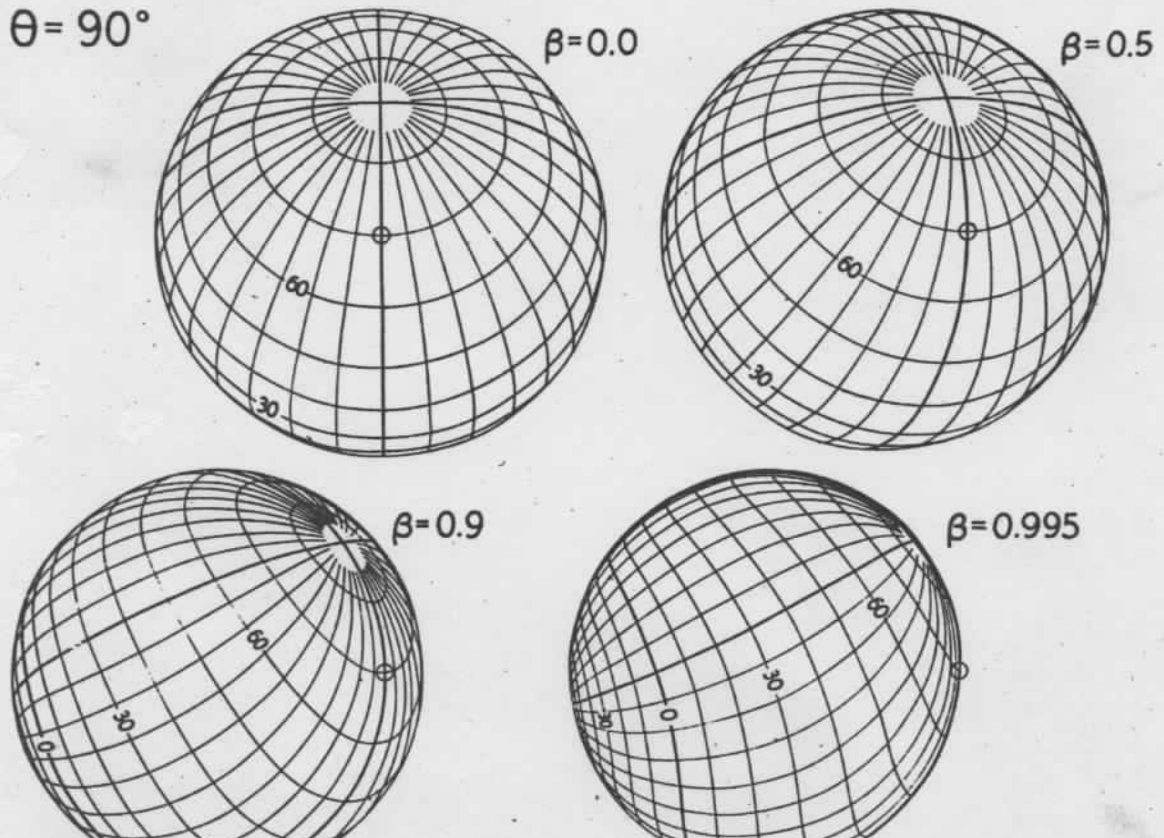


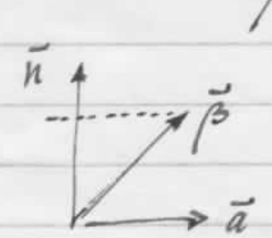
FIG. 5. Appearance of a sphere passing an observer. The conditions are as in Fig. 4 but the angle of observation θ is equal to 90°.

150191

So that the entire time is

$$\delta t + \beta \delta t \cos\psi = \delta t\left(1+\beta\left(\vec{n}\cdot\frac{\vec{\beta}}{\beta}\right)\right) = \delta t\left(1+\vec{n}\cdot\vec{\beta}\right) \qquad (45)$$

The transverse position vector is



$$\vec{a} = \vec{\beta} - (\vec{\beta}\cdot\vec{n})\,\vec{n} \qquad (46)$$

The vector identity $(\vec{A}\times\vec{B})\times\vec{C} = (\vec{A}\cdot\vec{C})\vec{B} - (\vec{B}\cdot\vec{C})\vec{A}$
gives

$$(\vec{\beta}\cdot\vec{n})\,\vec{n} = (\vec{\beta}\times\vec{n})\times\vec{n} - \vec{\beta} \qquad (47)$$

so that $\quad \vec{a} = \vec{n}\times(\vec{\beta}\times\vec{n}) \qquad (48)$

and finally

$$\vec{\beta}_{obs} = \frac{\vec{n}\times(\vec{\beta}\times\vec{n})}{1+\vec{\beta}\cdot\vec{n}} \qquad (49)$$

NB: Blandford & Königl, ('79 ApJ 232, 34), use $\vec{n}$ to be the vector from source to observer; so they get

$$\vec{\beta}_{obs} = \frac{\vec{n}\times(\vec{\beta}\times\vec{n})}{1-\vec{\beta}\cdot\vec{n}} \qquad (50)$$

Note that β_{obs} can approach infinity! In the denominator we recognise the Doppler & aberration factor $1-\beta\cos\theta$.