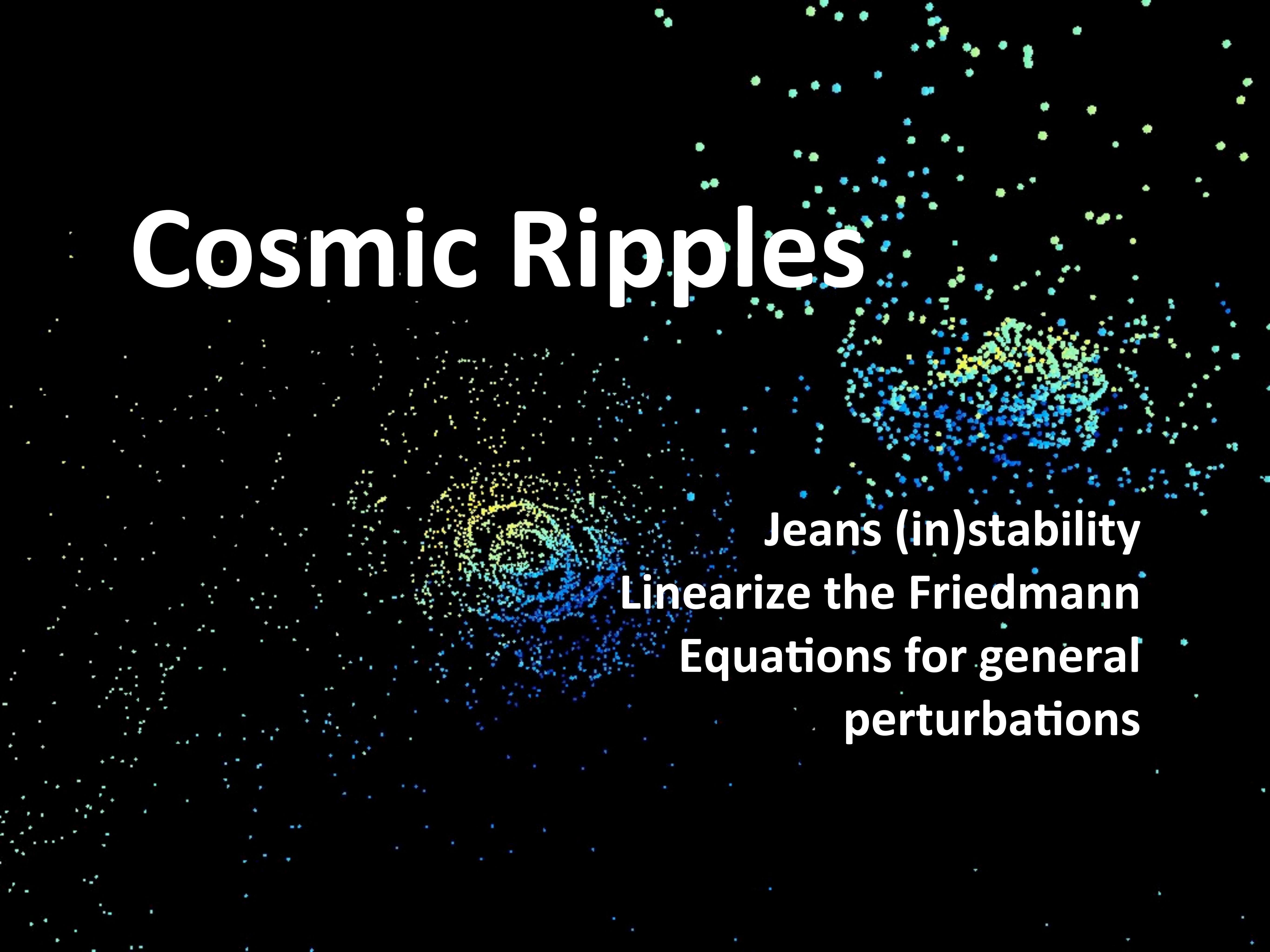


Cosmic Ripples

Jeans (in)stability
Linearize the Friedmann Equations for general perturbations

The second equation can be substituted into the first, to yield finally

$$\frac{d^2\delta}{dt^2} + 2H\frac{d\delta}{dt} + \left(\frac{s^2q^2}{a^2} - 4\pi G\rho\right)\delta = 0 \qquad (1.14)$$

Substitution of the value of ρ in an E-dS universe produces the corollary

$$\frac{d^2\delta}{dt^2} + 2H\frac{d\delta}{dt} + \left(\frac{s^2q^2}{a^2} - \frac{3}{2}H^2\right)\delta = 0 \qquad (1.15)$$

The second term is due to damping by adiabatic expansion (the same term we encountered earlier in the comoving velocity); the third term describes oscillation or growth of the perturbation. Obviously, oscillatory behaviour is determined by the sign of the expression in brackets: if this is negative, we get a runaway solution. The dividing line between oscillation and growth is then found by putting this term to exactly zero. This gives

$$q = \left(\frac{4\pi G \rho a^2}{s^2} \right)^{1/2} \propto \left(\frac{GM/a}{s^2} \right)^{1/2} \propto \frac{v_{\rm free}}{s} \qquad (1.16)$$

This is the famous `Jeans condition`: if a perturbed mass can fall freely more rapidly than pressure waves can travel, then counterpressures inside always come too late to stabilize the perturbed mass against gravitational collapse.

Let us suppose that we are well above the Jeans limit given by this condition. Then the pressure term (i.e. the one proportional to s^2) can be neglected, and the perturbation equation becomes

$$\frac{d^2\delta}{dt^2} + \frac{4}{3t}\frac{d\delta}{dt} - \frac{2}{3t^2}\delta = 0 \tag{1.17}$$

in which we have assumed an Einstein-De Sitter background. This equation has two solutions:

$$\delta \propto t^{2/3}\ ; \qquad \delta \propto t^{-1} \tag{1.18}$$

We saw that small density perturbations obey

$$\frac{d^2\delta}{dt^2} + 2H\frac{d\delta}{dt} + \left(\frac{s^2q^2}{a^2} - \frac{3}{2}H^2\right)\delta = 0 \tag{2.1}$$

Well above the Jeans limit, we can use $s \approx 0$, so that we get

$$\begin{aligned} &\frac{d^2\delta}{dt^2} + 2H\frac{d\delta}{dt} - \frac{3}{2}H^2\delta = 0 \qquad \text{(nonrelativistic)} \\ &\frac{d^2\delta}{dt^2} + 2H\frac{d\delta}{dt} - 4H^2\delta = 0 \qquad \text{(relativistic)} \end{aligned} \tag{2.2}$$

(for the relativistic case see Weinberg p.588). Inserting the value of H in the two cases, one obtains

$$
\begin{aligned}
\frac{d^2\delta}{dt^2} + \frac{4}{3t}\frac{d\delta}{dt} - \frac{2}{3t^2}\delta = 0 \\
\frac{d^2\delta}{dt^2} + \frac{1}{t}\frac{d\delta}{dt} - \frac{1}{t^2}\delta = 0
\end{aligned}
\tag{2.3}
$$

The growing solutions are

$$
\delta \propto \begin{cases} t^{2/3} \\ t \end{cases}
\tag{2.4}
$$

Below the critical Jeans mass we have the other limiting case of the perturbation equation, namely

$$\frac{d^2\delta}{dt^2} + 2H\frac{d\delta}{dt} + \frac{s^2q^2}{a^2}\delta = 0 \tag{2.6}$$

This equation has constant amplitude oscillations in the relativistic regime, whereas after t_e we find oscillations with a slowly decreasing amplitude proportional to $t^{-1/6}$. In the Einstein-De Sitter case we have $H = 2/3t = 2/3t_0\tau$, so that

$$\frac{d^2\delta}{d\tau^2} + \frac{4}{3\tau}\frac{d\delta}{d\tau} + s^2q^2t_0^2\frac{\delta}{a^2} = 0 \tag{2.7}$$

$$\frac{d^2\delta}{d\tau^2} + \frac{4}{3\tau}\frac{d\delta}{d\tau} + s^2q^2t_0^2\frac{\delta}{a^2} = 0 \qquad (2.7)$$

The difficulty is the behaviour of s^2q^2; because the Universe expands, the speed of sound s is not a constant. For a given wave number q, the effective propagation speed changes due to the changing thermal conditions (mostly adiabatic expansion, except during exothermic episodes of change).

According to the Friedmann Equations, the square of a velocity (or an energy) evolves according to $v^2 \propto \frac{1}{a}$. Or else, knowing that s^2 is proportional to the temperature, we can use Wien's Law: $s^2 \propto T \propto 1/a$. Therefore we may write $s^2 = \frac{s_0^2}{a}$ and we get

$$\frac{d^2\delta}{d\tau^2} + \frac{4}{3\tau}\frac{d\delta}{d\tau} + s_0^2 q^2 t_0^2 \frac{\delta}{a^3} = \frac{d^2\delta}{d\tau^2} + \frac{4}{3\tau}\frac{d\delta}{d\tau} + \frac{p^2}{\tau^2}\delta = 0 \quad (2.8)$$

$$\frac{d^2\delta}{d\tau^2} + \frac{4}{3\tau}\frac{d\delta}{d\tau} + s_0^2 q^2 t_0^2 \frac{\delta}{a^3} = \frac{d^2\delta}{d\tau^2} + \frac{4}{3\tau}\frac{d\delta}{d\tau} + \frac{p^2}{\tau^2}\delta = 0 \quad (2.8)$$

The solution of this equation can be found by noting that the whole equation scales as the inverse second power of time, which immediately suggests the *Ansatz* $\delta \propto \tau^\alpha$. This leads to a quadratic equation for α,

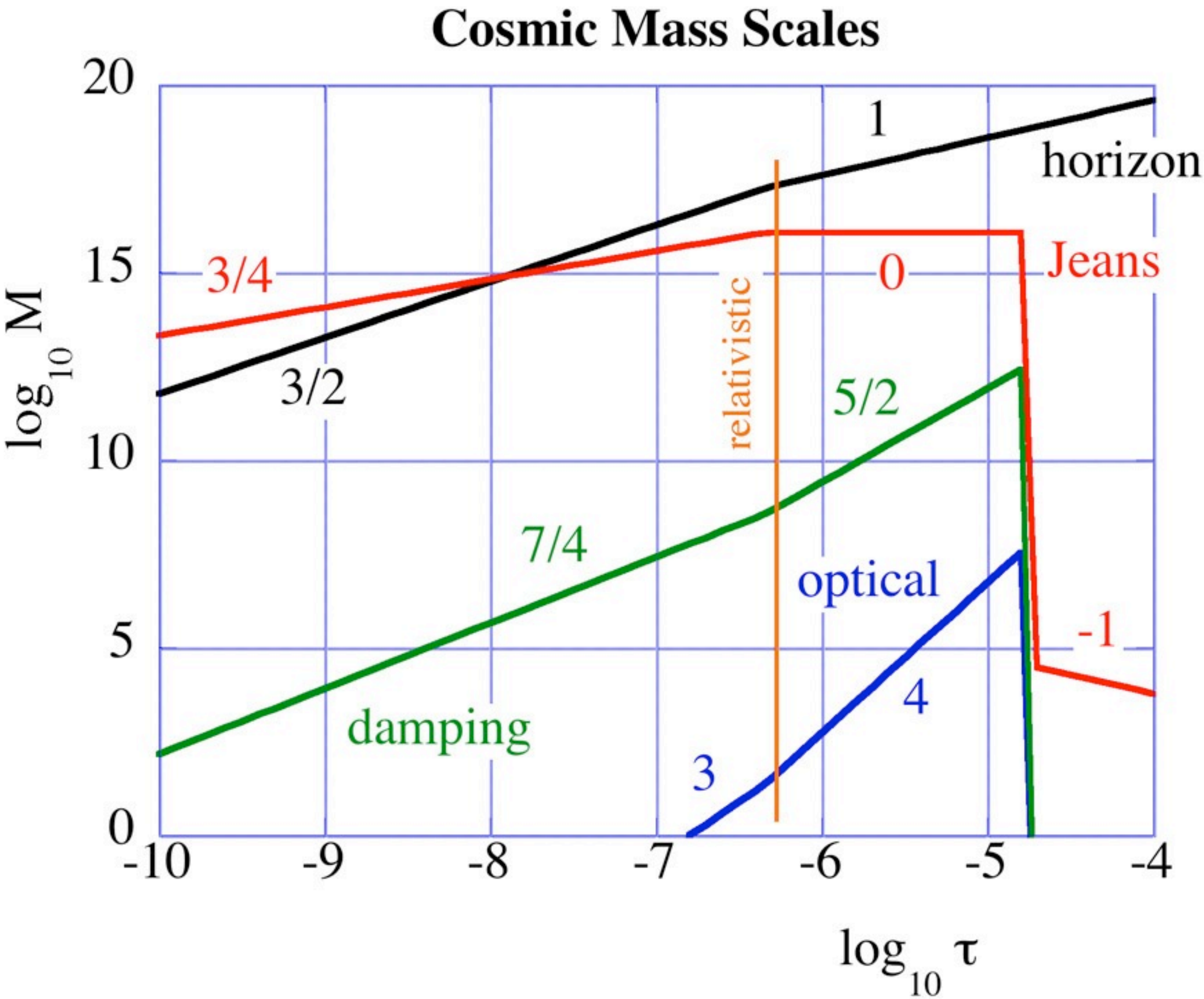
$$\alpha(\alpha - 1) + \frac{4}{3}\alpha + p^2 = 0\ ; \qquad p^2 \equiv \frac{1}{36} + \omega^2 \quad (2.9)$$

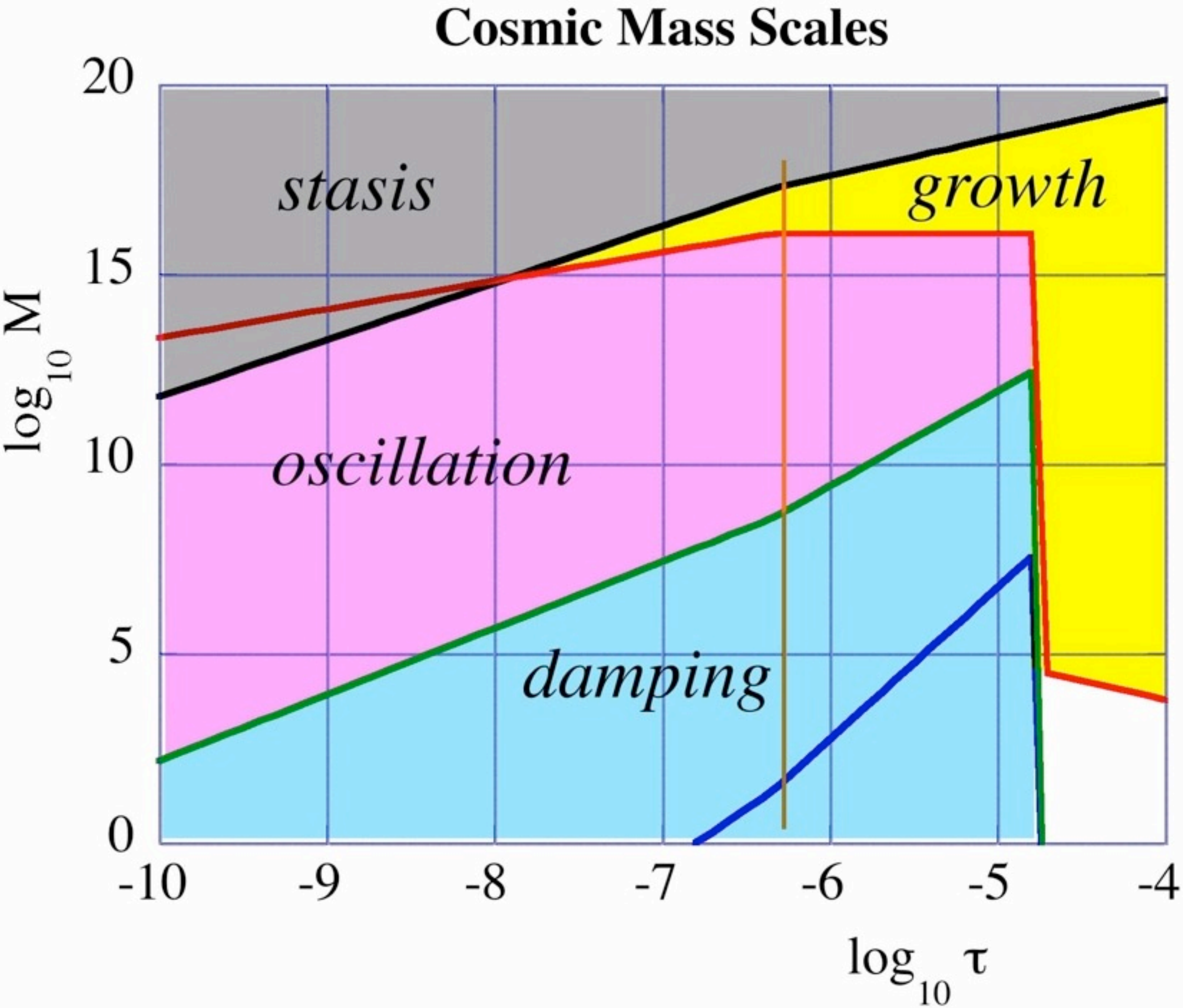
which has the solutions

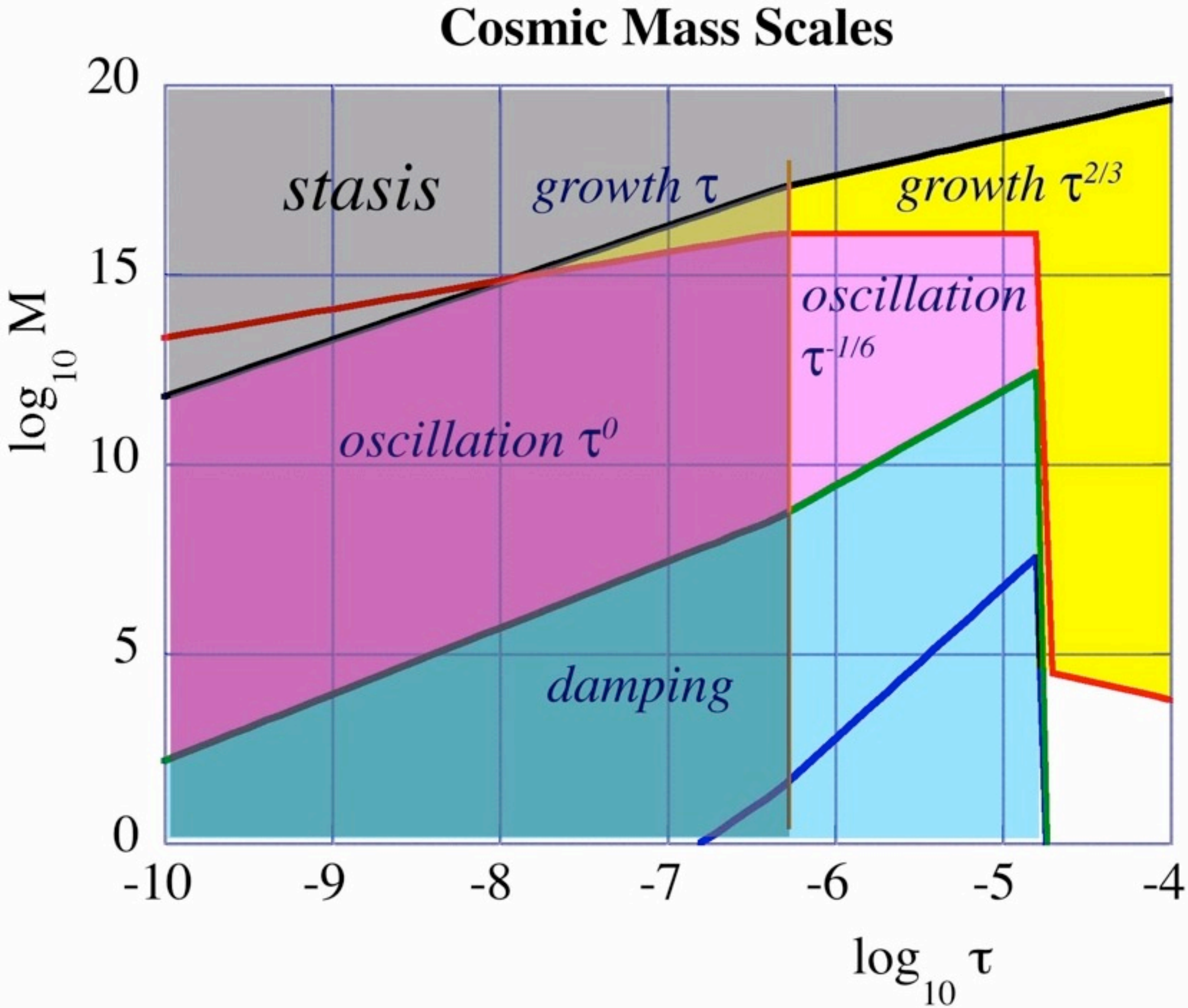
$$\alpha = -\frac{1}{6} \pm i\omega \quad (2.10)$$

Hence, the overall solution is

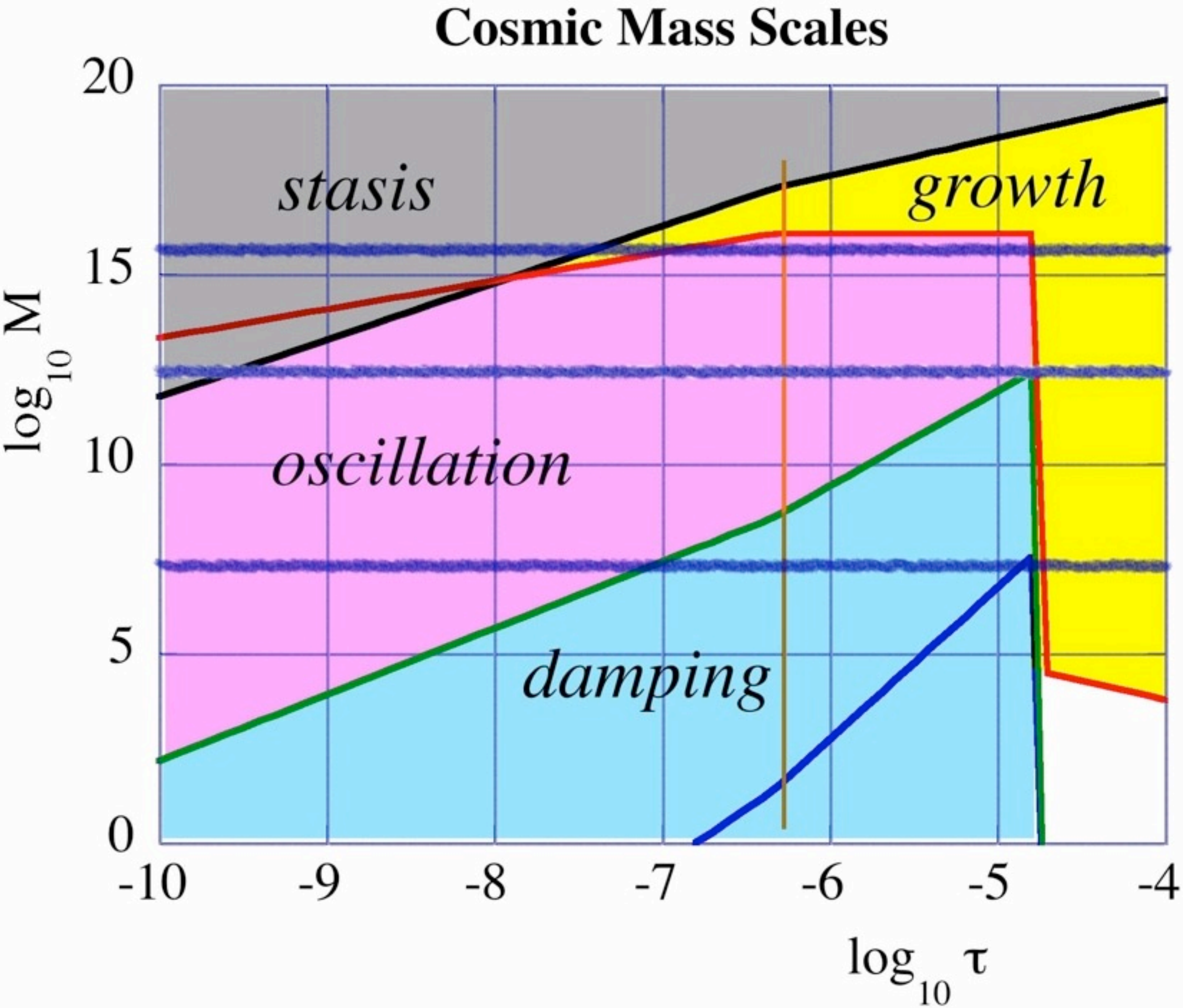
$$\delta = \delta_0 \tau^{-1/6} \cos(\omega \log \tau + \phi); \quad \tau \equiv \frac{t}{t_0} \quad (2.11)$$


Cosmic Mass Scales
20
15
10
5
0
-10
-9
-8
-7
-6
-5
-4
log10 M
log10 τ
1
horizon
3/4
0
Jeans
3/2
relativistic
5/2
7/4
optical
-1
4
damping
3


Cosmic Mass Scales
20
15
10
5
0
log10 M
-10
-9
-8
-7
-6
-5
-4
log10 τ
stasis
growth
oscillation
damping

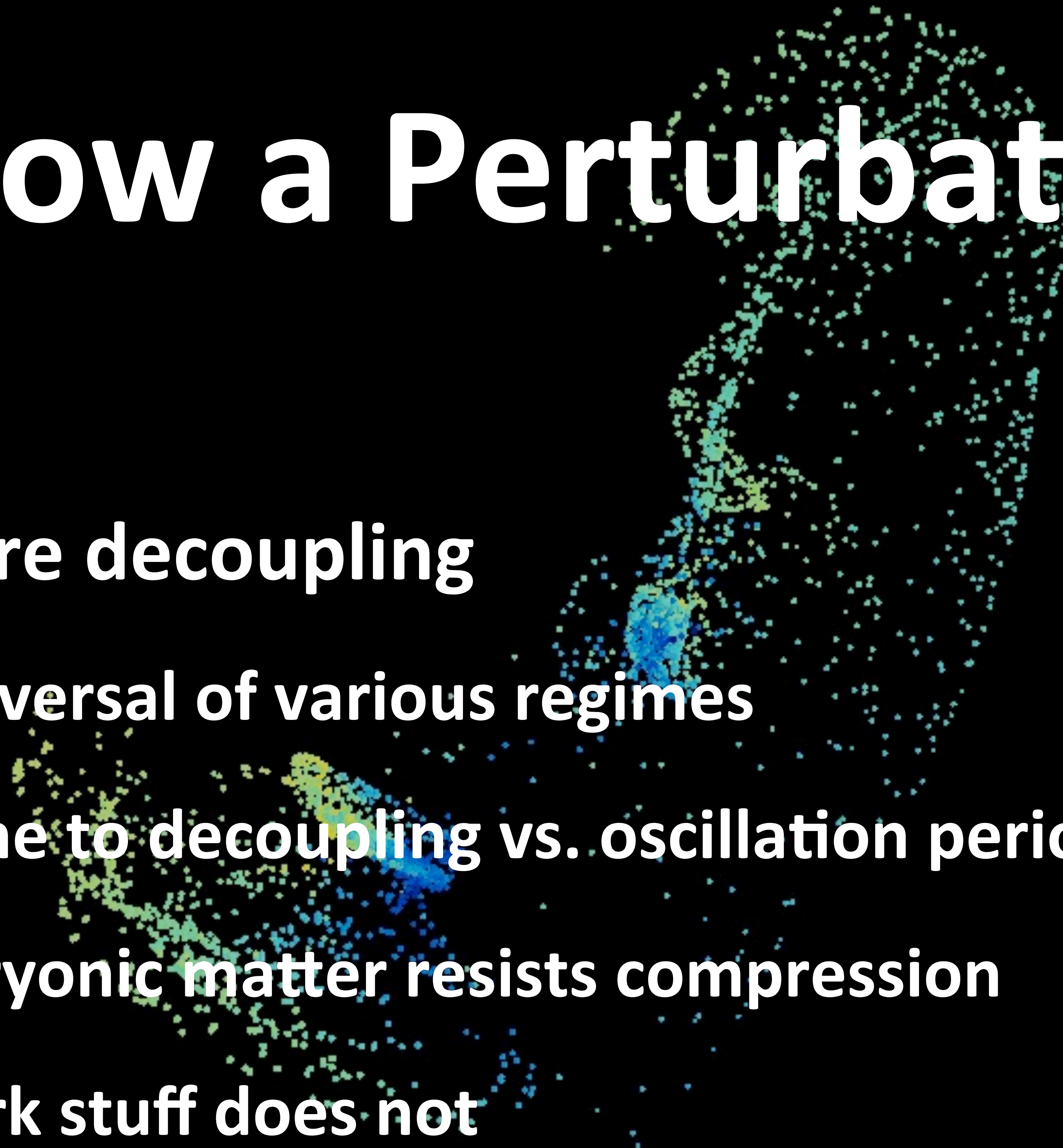


Cosmic Mass Scales
20
15
10
5
0
-10
-9
-8
-7
-6
-5
-4
log10 M
log10 τ
stasis
growth τ
growth τ2/3
oscillation τ-1/6
oscillation τ0
damping



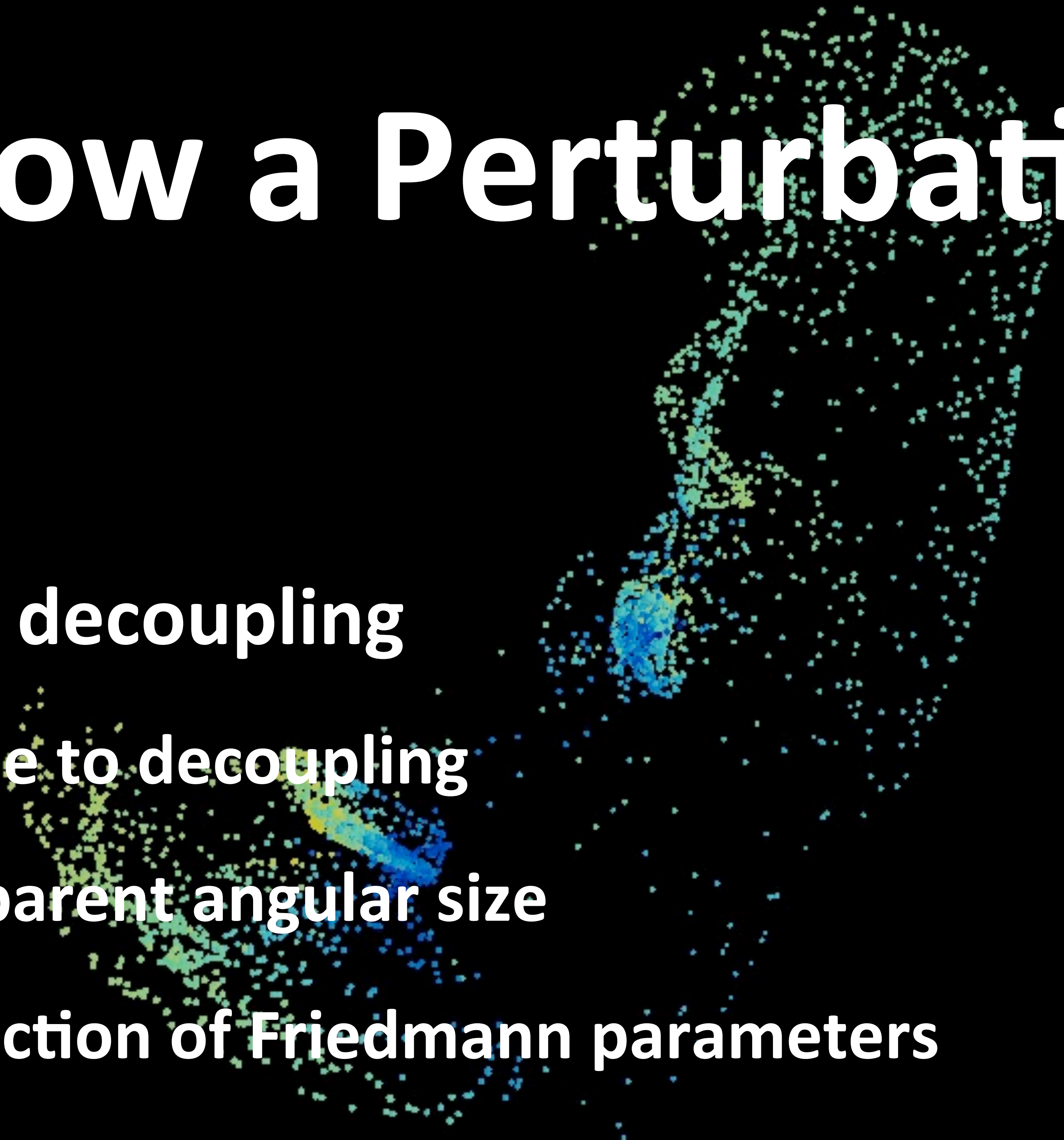
Cosmic Mass Scales
20
15
10
5
0
log10 M
stasis
growth
oscillation
damping
-10
-9
-8
-7
-6
-5
-4
log10 τ

Follow a Perturbation

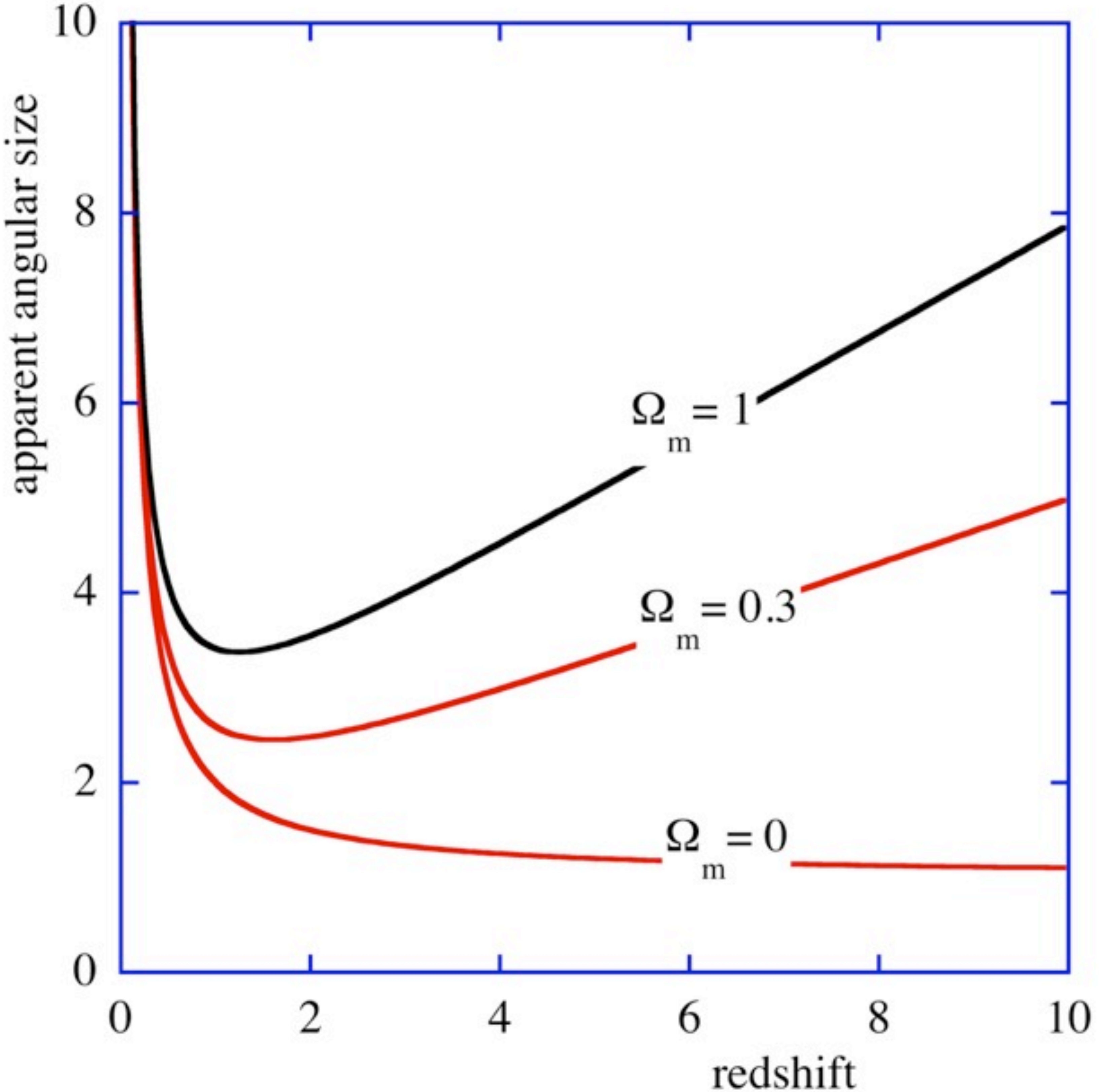


- Before decoupling
  - Traversal of various regimes
  - Time to decoupling vs. oscillation period
  - Baryonic matter resists compression
  - Dark stuff does not

Follow a Perturbation



- After decoupling
  - Time to decoupling
  - Apparent angular size
  - Function of Friedmann parameters

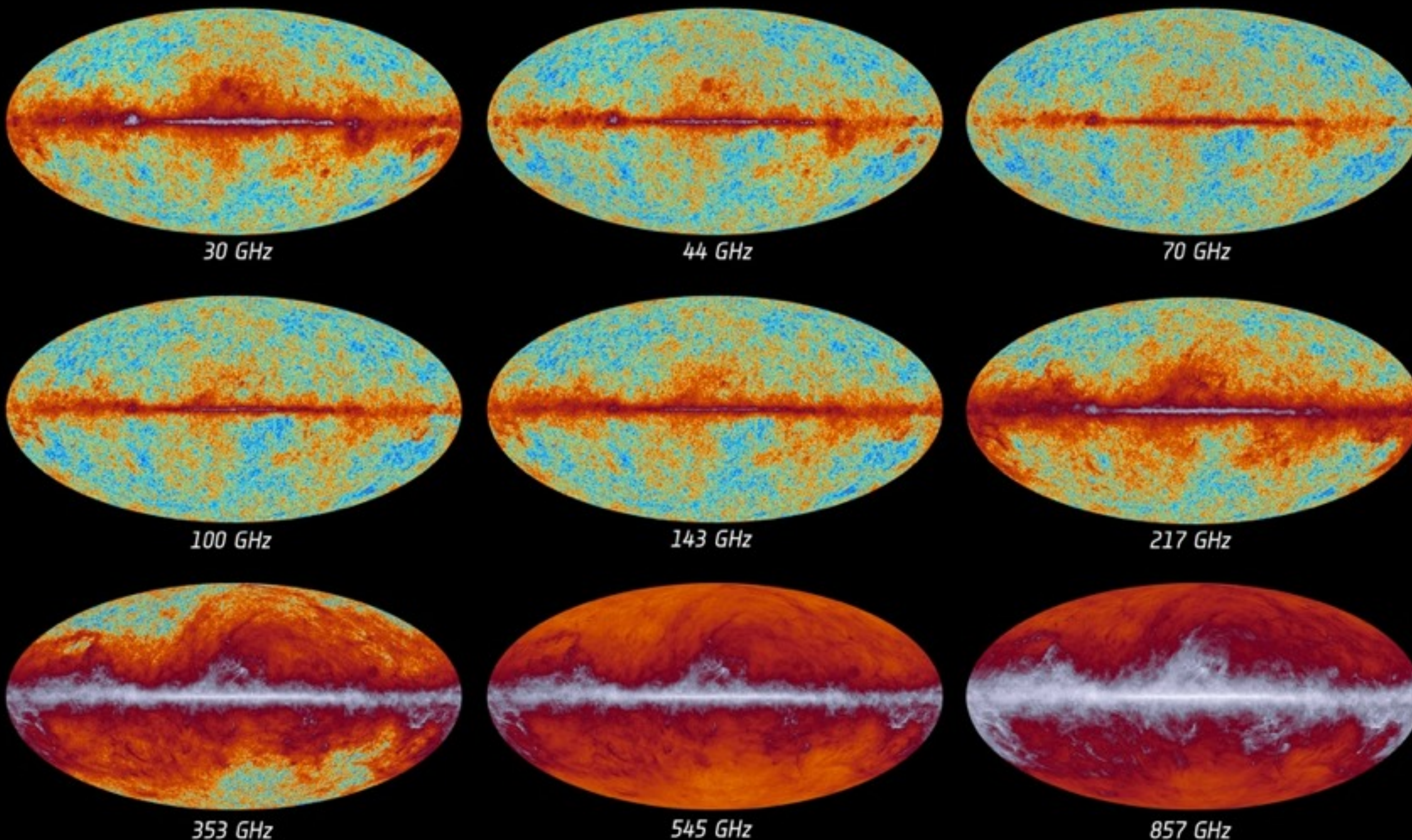


apparent angular size
redshift
0
2
4
6
8
10
Ω m = 1
Ω m = 0.3
Ω m = 0

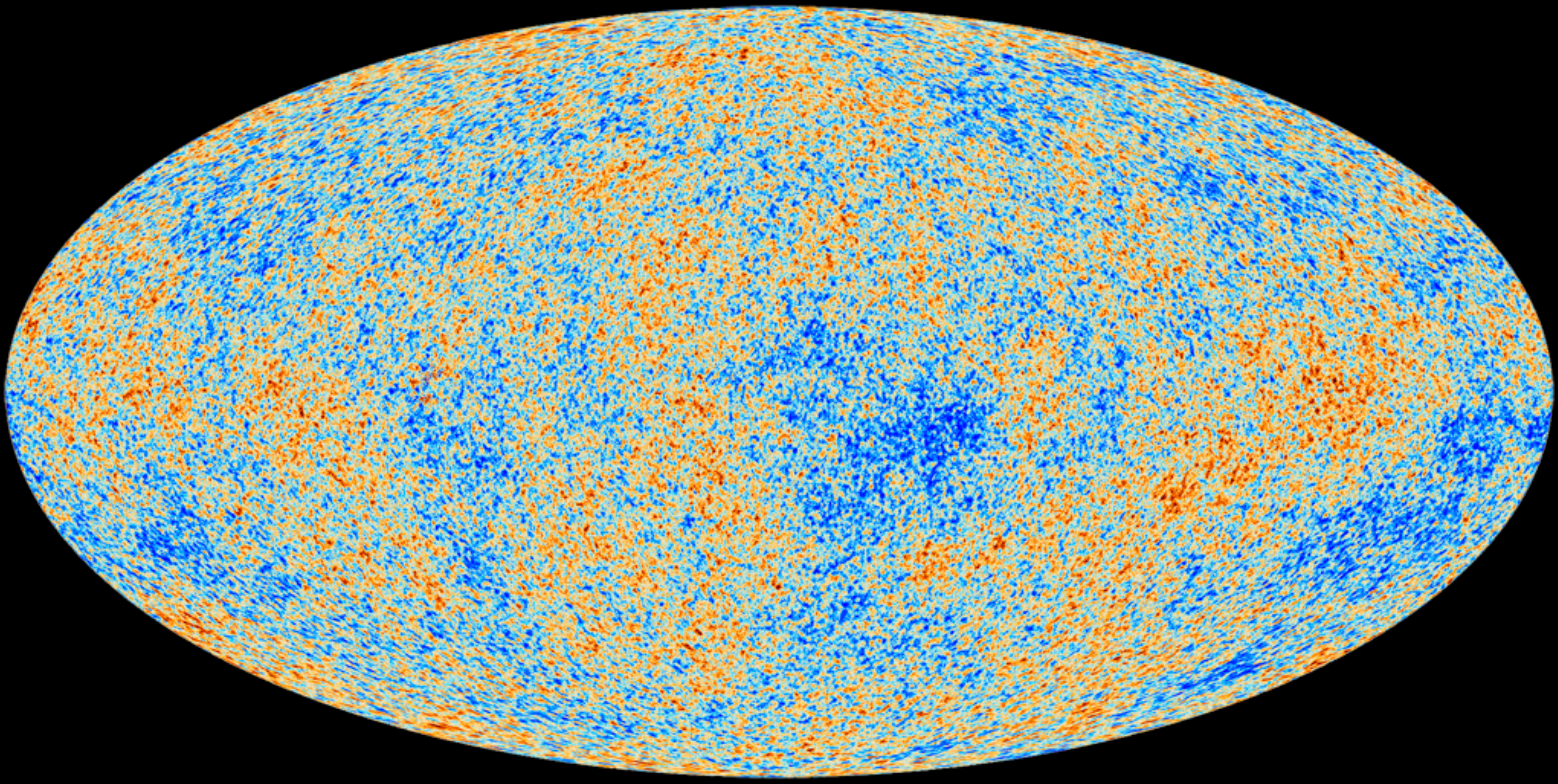
planck

The sky as seen by Planck

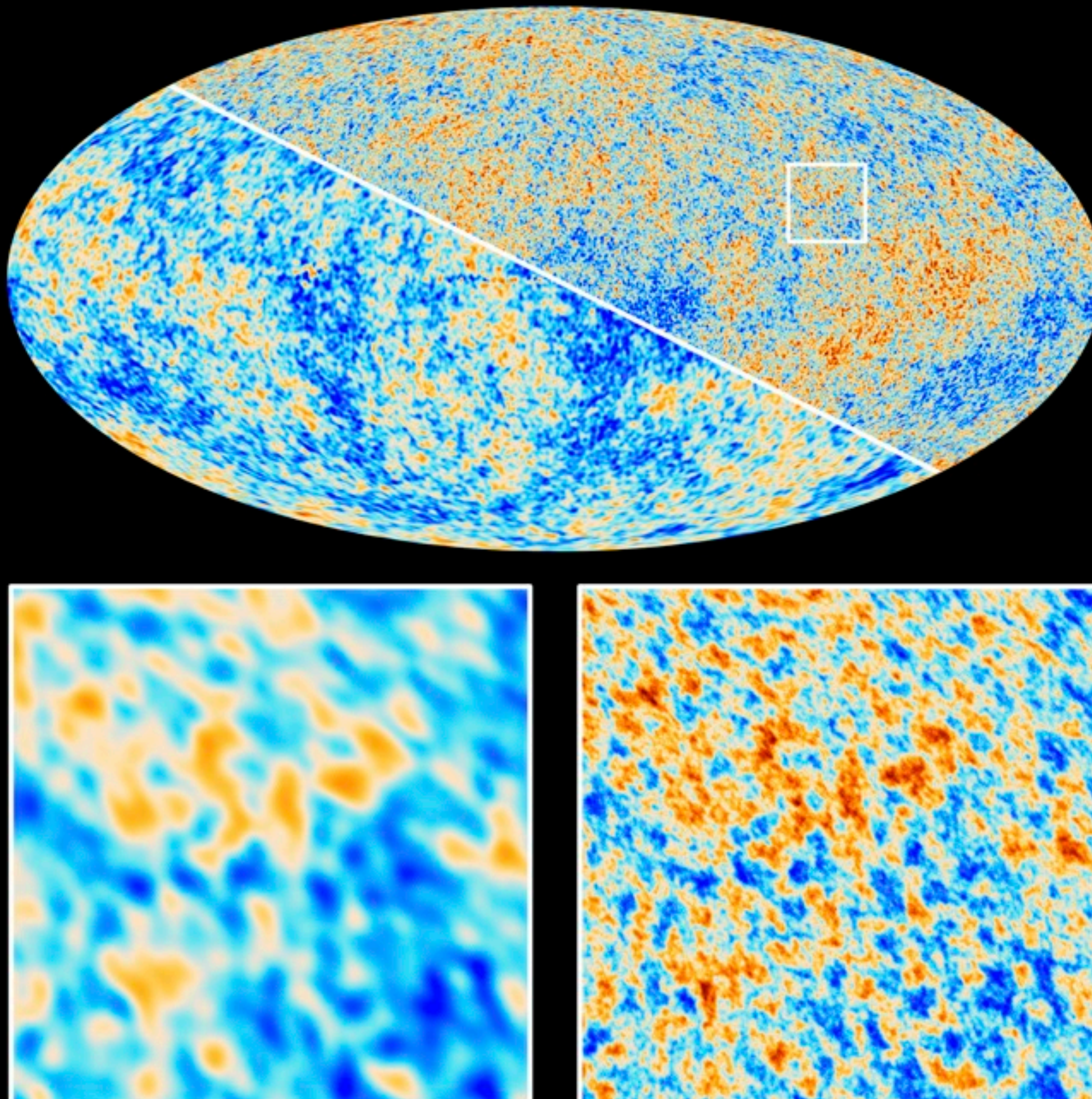
esa



Corrections for foreground emission

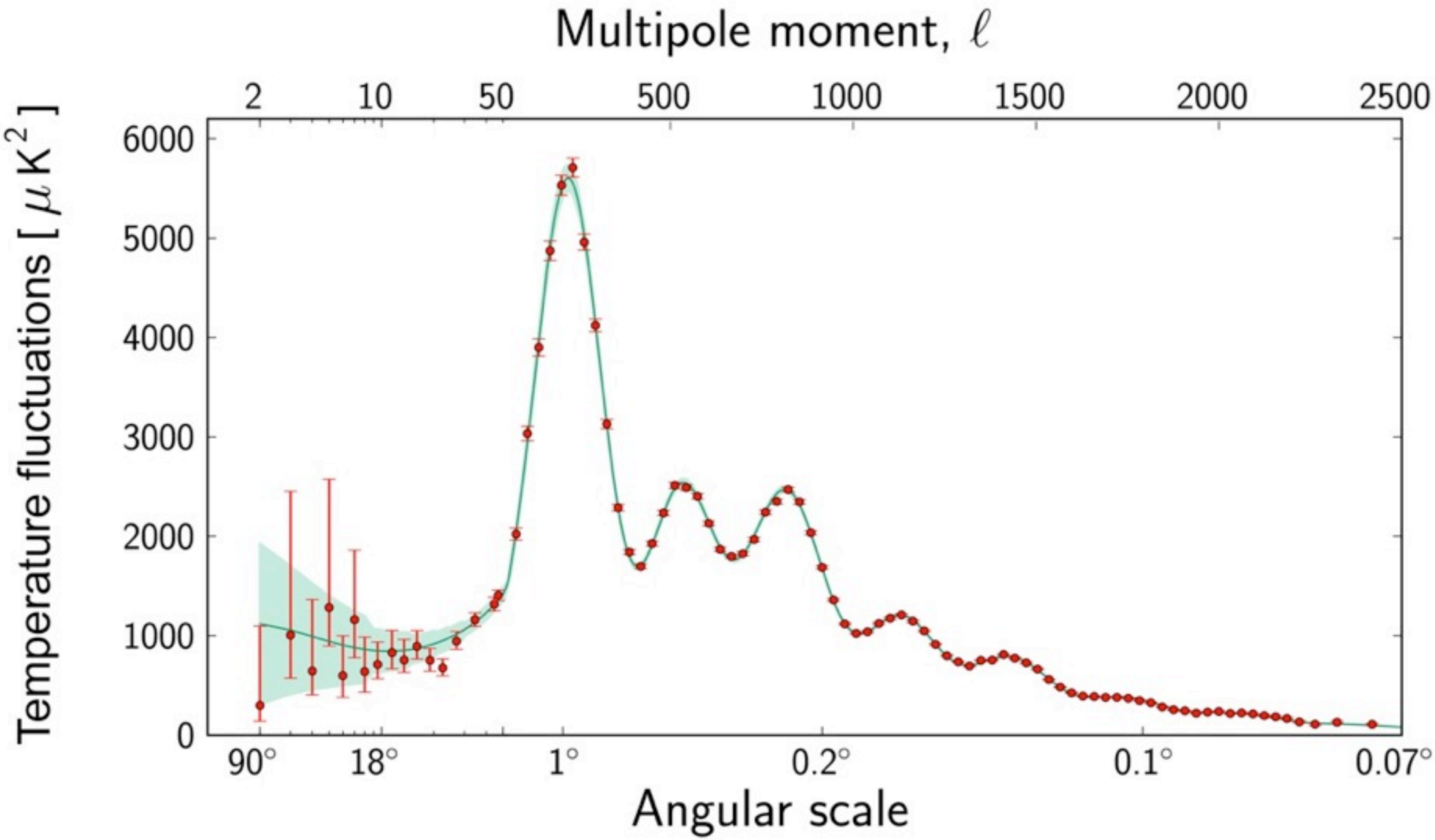


The Cosmic Microwave Background as seen by Planck and WMAP



WMAP　　　　Planck

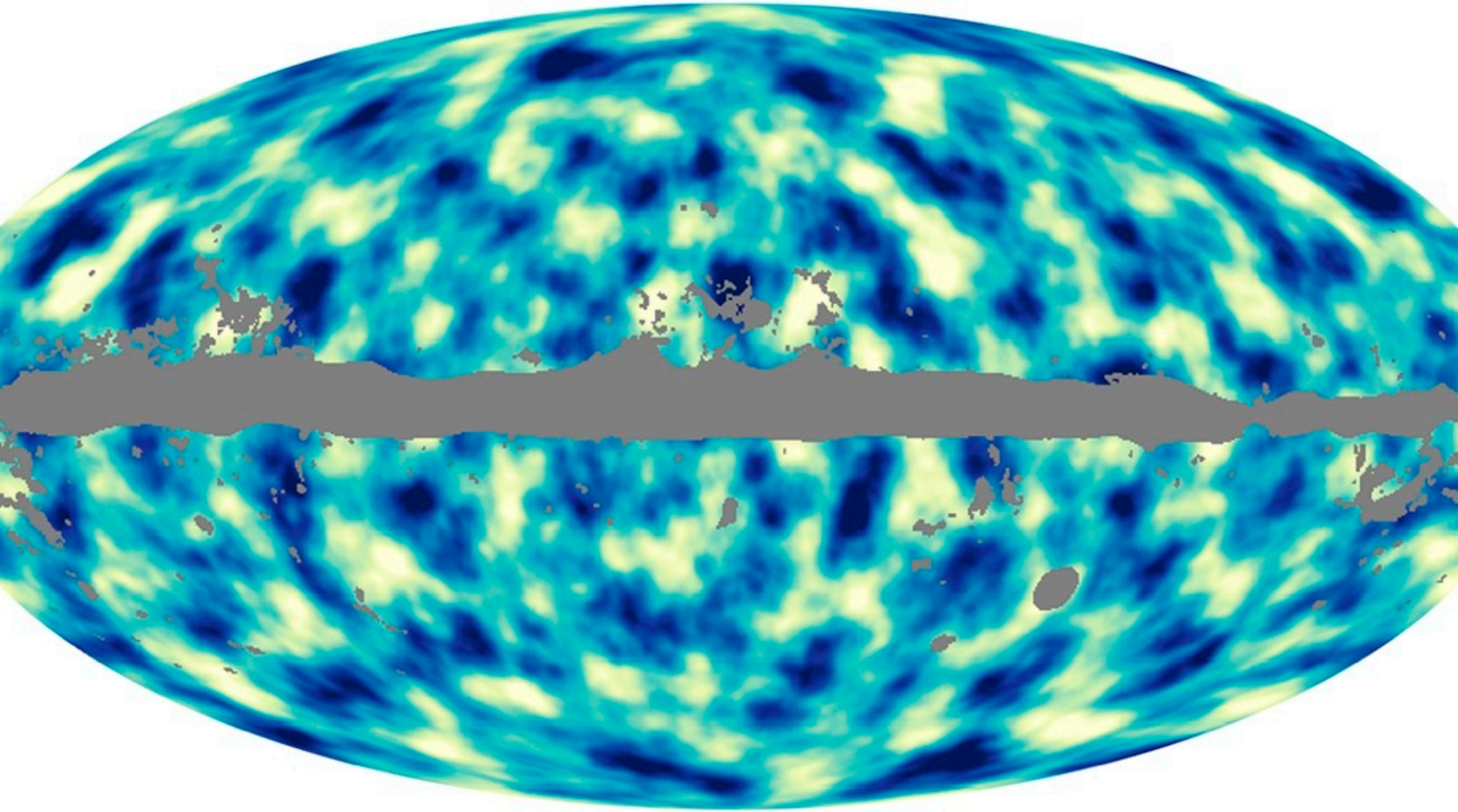
Multipole decomposition (power spectrum)


Multipole moment, ℓ
2
10
50
500
1000
1500
2000
2500
Temperature fluctuations [μK²]
6000
5000
4000
3000
2000
1000
0
90°
18°
1°
0.2°
0.1°
0.07°
Angular scale

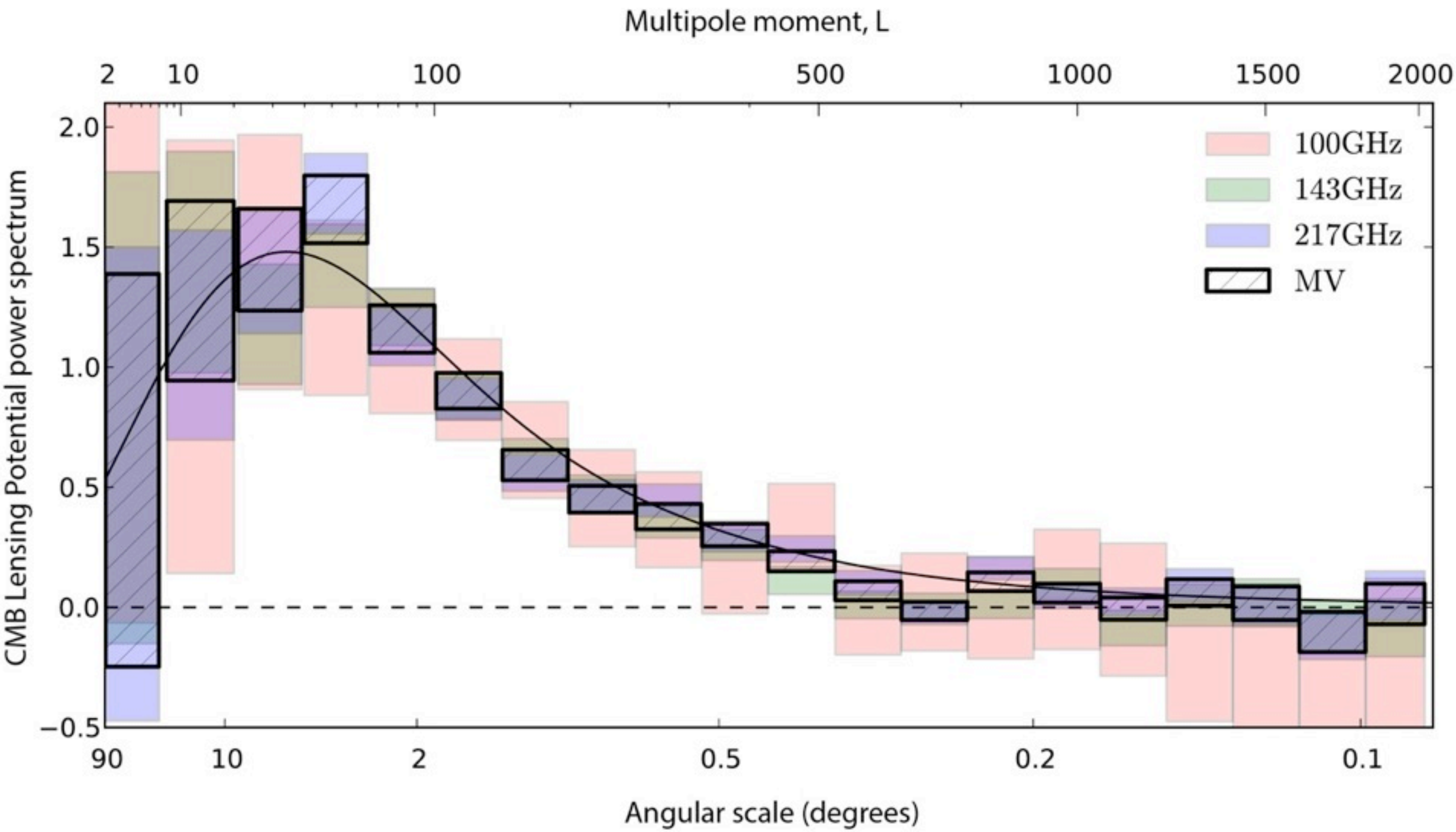
Systematic differences between model and observations

Influence of the gravitational lens effect

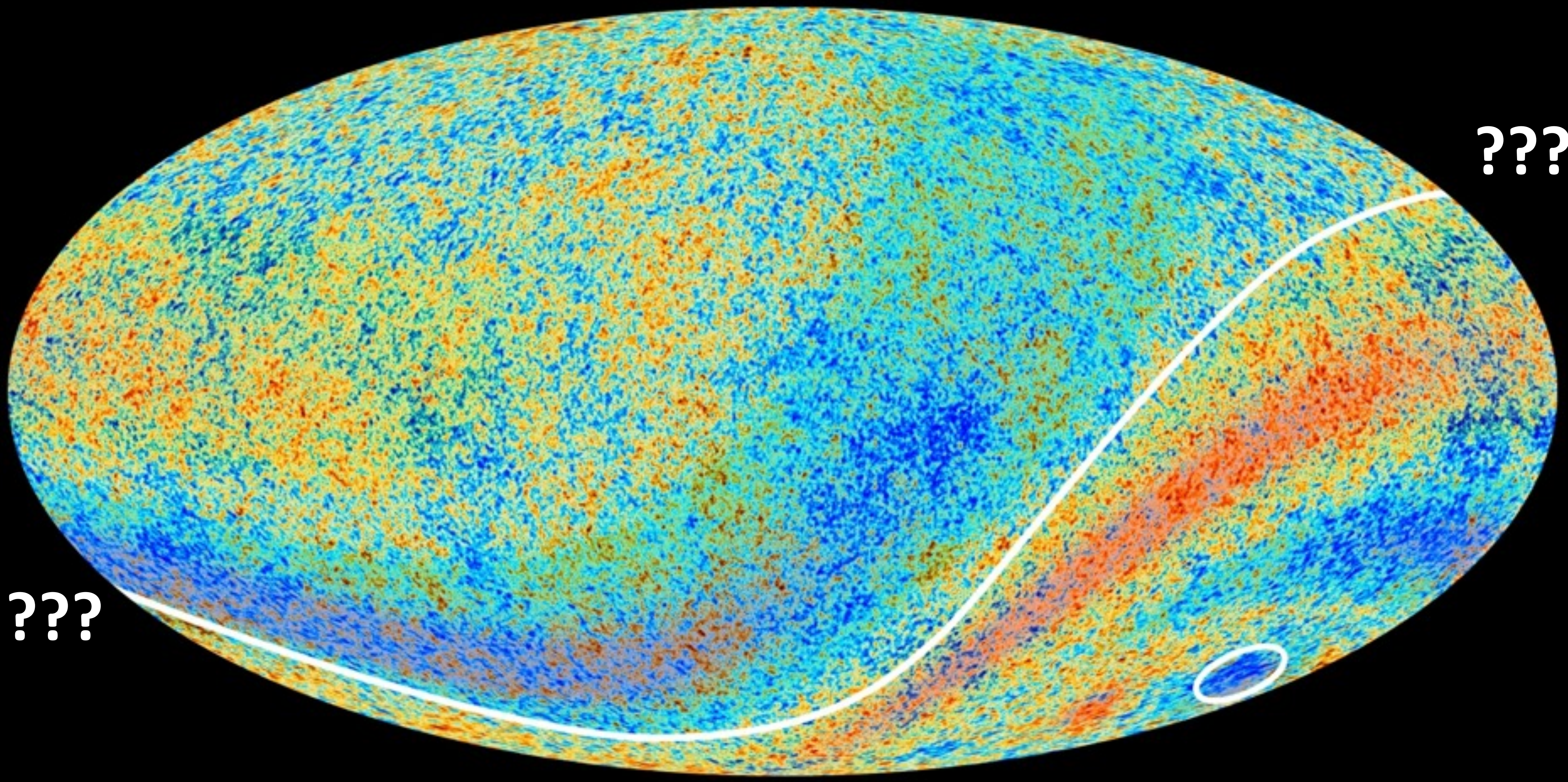
Dark Stuff distribution



Dark Stuff power spectrum



Unexplained residuals



Contributions of the first and third Friedmann term