**ORIGINS AND EVOLUTION OF THE UNIVERSE - PROBLEM SET #3**

**DUE WEDNESDAY DECEMBER 5, 2018**

This is the third problem set for this course. The solutions to the problems in this set will be discussed on Wednesday December 5 from 13:30 in HL414. Your solutions will be graded if they are received at or before that time. There are six exercises in this set.

1. Consider a scalar field $\phi$ with density $\rho = \frac{1}{2}\dot{\phi}^2 + V(\phi)$ and pressure $p/c^2 = \frac{1}{2}\dot{\phi}^2 - V(\phi)$.
   (a) (*4 points*) Use the fluid equation to show that the evolution of this scalar field obeys:

$$\ddot{\phi} + 3\frac{\dot{a}}{a}\dot{\phi} + \frac{\partial V}{\partial \phi} = 0.$$

   In the slow roll approximation we assume that $\ddot{\phi}$ is negligible compared to the other two terms in the equation that describes the evolution of $\phi$. Furthermore, the kinetic term is assumed to be small compared to the potential energy, i.e., $\frac{1}{2}\dot{\phi}^2 \ll V(\phi)$.
   (b) (*4 points*) Consider a scalar field theory with a quadratic potential $V(\phi) = m^2\phi^2$. Using the equation you derived in part (a) and the Friedmann equation, derive the time evolution of this scalar field $\phi(t)$ and the scale factor $a(t)$ in the slow roll approximation. The requirements for slow roll can be quantified using the slow roll parameters $\epsilon$ and $\eta$, which are functions of the first and second derivatives of the potential. The condition $\epsilon \ll 1$ ensures exponential expansion, whereas $\eta \ll 1$ ensures that the inflationary periods lasts sufficiently long.
   (c) (*4 points*) Assuming slow roll, derive the expression relating $\dot{\phi}^2$ to $V' = \frac{\partial V}{\partial \phi}$ and $V$. Use this expression to show that the assumption that $\frac{1}{2}\dot{\phi}^2 \ll V(\phi)$ is equivalent to assuming that

$$\epsilon \equiv \frac{1}{48\pi G}\left(\frac{V'}{V}\right)^2 \ll 1.$$

   (d) (*4 points*) For inflation to last, $\ddot{\phi}$ should be small, so we stay in the slow roll regime. Show that

$$\ddot{\phi} = -\frac{1}{3H}V''\dot{\phi} + \frac{1}{H}\frac{V'^2}{6V}\dot{\phi}$$

   (e) (*4 points*) Show that the assumption $\ddot{\phi} \ll V'$ is equivalent to the assumption that

$$\eta \equiv \frac{1}{24\pi G}\frac{V''}{V} \ll 1.$$

2. In the standard model of particle physics neutrinos have no mass and are therefore, like photons, still relativistic.
   (a) (*5 points*) By considering entropy conservation, derive the present day temperature of the neutrino background.
   (b) (*5 points*) Derive the present day density parameter $\Omega_{\nu,0}$ of these neutrinos, under the assumption that there are three families of neutrinos. How does this compare to the present density in photons?
   (c) (*4 points*) Express the number density of a neutrino species in terms of the number density of photons, just before the annihilation of electrons and positrons. How does the number density of photons change by the annihilation of electrons and positrons?

   Observations of neutrino oscillations imply that neutrinos are not massless. The actual masses are still unknown, but better observations of the growth of large-scale structure may be able to determine the masses in the coming years.
   (d) (*4 points*) Suppose that one of the neutrino families has a non-zero mass $m_\nu = 0.1$ eV, so that it was relativistic when it decoupled, but is now non-relativistic. What is the present day density parameter of these neutrinos? How does this compare to the density of relativistic neutrinos calculated in part (b)?

3. Before the weak force decouples the number densities of protons and neutrons are in thermal equilibrium. In this exercise we assume that the weak force decouples at a temperature of $k_B T = 1.4$ MeV. The mass difference between a neutron and a proton is 1.29 MeV/$c^2$.
   (a) (*5 points*) What is the ratio of neutrons to protons when the weak force decouples?

   (b) (*5 points*) The protons and neutrons begin forming $^4$He at time $t = 200$s. If the half-life of neutrons is $\tau_{1/2} = 600$ s, derive the ratio of neutrons to protons at this time. What is the mass fraction of $^4$He that is produced during BBN?

   (c) (*4 points*) Calculate the mass fraction of $^4$He if the half life of neutrons is doubled. Comment on the result.

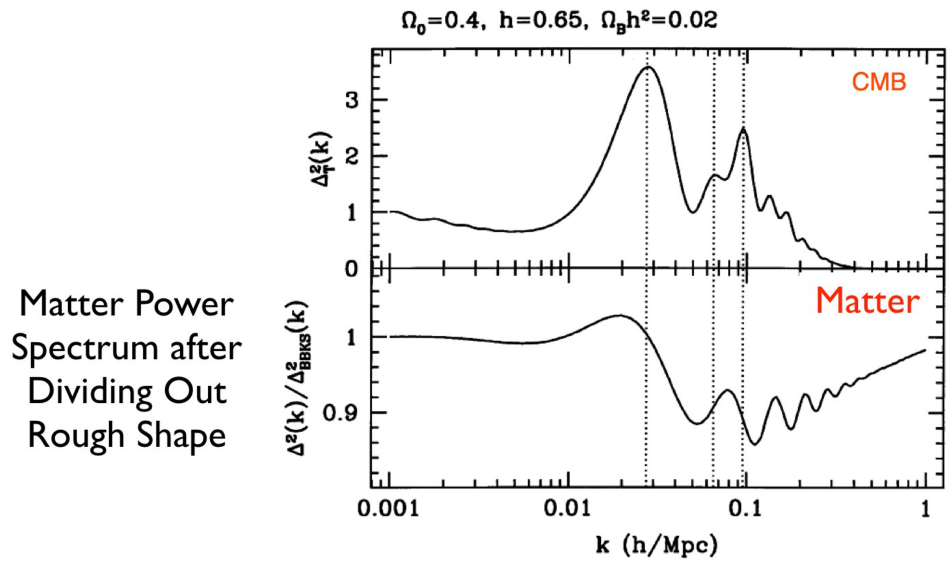
4. The largest scales that can leave a feature in the power spectrum correspond to the horizon at the epoch when such a feature is frozen.
   (a) (*4 points*) Consider a mode which has a wavelength equal to the horizon at a given epoch. Assuming a flat radiation-dominated universe, express the comoving wavenumber $k$ in terms of $H_0$, $c$ and $z$. Do the same for a flat matter-dominated universe.
   (b) (*4 points*) At the present day, $\Omega_{m,0} = 0.27$, $\Omega_{rad,0} = 8 \times 10^{-5}$, and $H_0 = 69$ km/s/Mpc. Estimate the value of $k$ of this mode at the epoch of matter-radiation equality. Assuming there is an imprint left on that scale, which remains constant in comoving coordinates until the present day, what physical wavelength would this correspond to at $z = 0$?

   For a relativistic gas, the sound speed is given by $cs = c/\sqrt{3}$. Assume that during the radiation-dominated era, this relativistic gas is coupled to a non-relativistic baryon fluid with mass density $\rho_b$. In this case the sound speed is given by

$$c_s^2 = \frac{c^2}{3}\left(1 + \frac{3}{4}\frac{\rho_b}{\rho_r}\right)^{-1}$$

   (c) (*4 points*) Estimate the comoving distance a sound wave has propagated since the Big Bang during the radiation-dominated era (*Hint: to approximate the integral, recall that at high redshift the density in radiation is much larger than that in matter*).

5. (a) (*4 points*) Write a formula for the sound horizon, $d_H(z)$, which is the distance travelled by a sound wave starting at time $t = 0$ and finishing at a particular redshift z. Assume a sound speed $c_s = \mathrm{c}/\sqrt{3}$ in the photon-baryon plasma prior to recombination. Evaluate the size in Mpc of the sound horizon at $z_R = 1100$. Using the angular diameter distance (feel free to use the online cosmological calculator), calculate the angular size of this peak on the sky and the equivalent $l$ mode number for the first acoustic peak. Assume $\Omega_m = 0.25$, $\Omega_\Lambda = 0.75$. To simplify the problem, you can assume that the universe is matter-dominated throughout.
   (b) (*2 points*) To what comoving length scale does this region correspond at the present epoch ($z = 0$)? Note that the excess power on this length scale (which result from "baryon acoustic oscillations") can be probed using spectroscopic surveys for galaxies in the distance universe and effectively provides astronomers with a standard rod that can be used to derive constraints on the nature of dark energy.
   (c) (*6 points*) The power spectrum of galaxies in the local universe and the cosmic microwave background both provide us with probes of the same underlying set of primordial density fluctuations. Determine the $k$ mode in the matter power spectrum that corresponds to the equivalent multipole $l$ in the CMB power spectrum. Determine the $k$ mode (and hence comoving length scale in the present day universe) that corresponds to an $l$ of 20, 100, and 1000. Draw approximate diagram with matter power spectrum and the CMB, one over top the other. See the diagram on the next page for a guide.




6. Consider perturbations $\delta(t)$ of the cold dark matter component in a flat Universe containing matter with a mean density $\rho_{bg}$ and a cosmological constant $\Lambda$.

(a) (4 points) Use the Friedmann and acceleration equations to show that

$$\dot{H} = -4\pi G\rho_{bg} \tag{1}$$

and

$$\Lambda = 3H^2 + 2\dot{H}, \tag{2}$$

where $H(t) = \dot{a}/a$ is the Hubble parameter at time $t$.

Starting from the equations of motion describing a collisional fluid, we can introduce perturbations about a homogeneously expanding background and linearise these equations. If we consider solutions of the form $\delta\rho = D(t)e^{i\vec{k}\cdot\vec{r}}$, where $D(t) = \rho_{bg}\delta(t)$, we obtain the linear growth equation in a homogeneously expanding background

$$\ddot{\delta} + 2\frac{\dot{a}}{a}\dot{\delta} + (c_s^2 k^2 - 4\pi G\rho_{bg})\delta = 0.$$

(b) (*4 points*) We can introduce a new variable $u \equiv \delta(t)/H(t)$. Using the linear growth equation, show that u satisfies:

$$\frac{d^2}{dt^2}(uH) + 2H\frac{d}{dt}(uH) + uH\frac{dH}{dt} = 0 \tag{3}$$

(c) (*4 points*) By expressing derivatives with respect to $t$ as derivatives with respect to redshift $z$, show that equation 3 can be written as:

$$\frac{d^2u}{dz^2} + \left(\frac{3}{H}\frac{dH}{dz} - \frac{1}{1+z}\right)\frac{du}{dz} + \frac{u}{H}\left[\frac{d^2H}{dz^2} + \frac{1}{H}\left(\frac{dH}{dz}\right)^2 - \frac{2}{1+z}\frac{dH}{dz}\right] = 0 \tag{4}$$

(d) (*4 points*) By differentiating equation 2, show that:

$$\frac{d^2H}{dz^2} + \frac{1}{H}\left(\frac{dH}{dz}\right)^2 - \frac{2}{1+z}\frac{dH}{dz} = 0 \tag{5}$$

Hence $u$ satisfies:

$$\frac{d^2u}{dz^2} + \left(\frac{3}{H}\frac{dH}{dz} - \frac{1}{1+z}\right)\frac{du}{dz} = 0 \tag{6}$$

(e) (*4 points*) Using equation 6, show that there are two solutions for the dark matter overdensity: a decaying mode $\delta(z) \propto H(z)$ and a growing mode:

$$\delta(z) = H(z)\int_z^\infty \frac{1+z'}{H^3(z')}dz' \tag{7}$$

(f) (*4 points*) If $\Lambda = 0$, show that these modes become: $\delta \propto a^{-3/2}$ and $\delta \propto a$.