ORIGINS AND EVOLUTION OF THE UNIVERSE - PROBLEM SET #2

DUE WEDNESDAY, NOVEMBER 14, 2018

This is the second problem set for this course. The solutions to the problems in this set will be discussed on Wednesdsay November 14 from 13:30 in HL414. Your solutions will be graded if they are received at or before that time. There are six exercises in this set.

1. During the period in which matter and radiation are decoupled, the matter temperature T_m and the radiation temperature T evolve independently of each other. If the matter component expands adiabatically in a homogeneous universe, and is assumed to be an ideal gas of only hydrogen, thermodynamics dictates that

$$d\left[\left(\rho_m c^2 + \frac{3\rho_m k_B T_m}{2m_p}\right) a^3\right] = -\frac{\rho_m k_B T_m}{m_p} d(a^3)$$

   (a) (*5 points*) Show that this implies that $T_m = T_{m,0}(1+z)^2$, where $T_{m,0}$ is the present-day temperature of the matter.

   Before decoupling the matter and radiation are tightly coupled. The expression that describes the adiabatic expansion of a gas of matter and radiation is

$$d\left[\left(\rho_m c^2 + \frac{3\rho_m k_B T}{2m_p} + \sigma_r T^4\right) a^3\right] = -\left(\frac{\rho_m k_B T}{m_p} + \frac{\sigma_r T^4}{3}\right) d(a^3)$$

   (b) (*5 points*) Show that this implies that

$$\frac{dT}{T} = -\frac{1 + \sigma_{rad}}{\frac{1}{2} + \sigma_{rad}} \frac{da}{a}$$

   where

$$\sigma_{rad} = \frac{4m_p \sigma_r T^3}{3k_B \rho_m}$$

2. We now consider the decoupling temperature of neutrinos, which can be estimated by comparing the typical interaction rate with the expansion rate H of the Universe. To get used to the literature, we set $c = \hbar = 1$ in this question. The cross-section for weak interactions depends on the momentum p (and hence temperature T) and is given by $\sigma \simeq G_F^2 p^2$, where $G_F = 1.17 \times 10^{-5}\,\mathrm{GeV}^{-2}$ is the Fermi coupling constant. With $c = \hbar = 1$ the number density of relativistic species is $n \simeq (k_B T)^3$ (for the temperatures we are interested in the coefficient happens to be close to unity).
   (a) (*5 points*) Assuming that the neutrinos are highly relativistic with a characteristic energy $k_B T$, obtain an expression for Γ, the interaction rate per neutrino in terms of the temperature. The Friedmann equation can be approximated as

$$H^2 = \frac{k_B^4 T^4}{(10^{19}\mathrm{GeV})^2}$$

   (b) (*5 points*) Use this to show that neutrinos decouple at a temperature of around 1 MeV.

3. Consider a universe with $\Omega_{m,0} = 3$, $\Omega_{\Lambda,0} = 0$, and $\Omega_{r,0} \sim 0$.
   (a) (*5 points*) Consider the propagation of a pulse of light from four sources with coordinate $(r, \theta, \phi) = (0, 0, 0)$ at $z = 0.2$, $z = 2$, $z = 20$, and $z = 2000$ (4 cases) to $z = 0$. By $z = 0$, the light will have propagated to different coordinates $r_{0.2}$, r_2, r_{20}, r_{2000}. What are those coordinates?
   (b) (*3 points*) If we represent the position of the light cone as a circle on the surface of a sphere (with coordinates θ and r), illustrate on the surface of a sphere (drawn or plotted) the coordinates where light from those sources would propagate by $z = 0$.
   (c) (*5 points*) How would you answer to part (a) change if you considered a universe with $\Omega_{m,0} = 2$ or 10? Are there values of $\Omega_{m,0}$ where light emitted by a source at $z = 2000$ would be reobserved by the same source at $z = 0$? Are there values of $\Omega_{m,0}$ where light emitted by a source at $z = 2000$ would reach the extreme opposite end of the universe by $z = 0$? If yes, what are they?
4. Typical Grand Unified Theories predict that magnetic monopoles with masses $M_M \sim 10^{15}\,\mathrm{GeV}/c^2$ will form in the early Universe when the temperature $k_B T_M \sim M_M c^2$. As discussed in class, on average one monopole will form per particle horizon volume at that time. In this question we will quantify the monopole problem. We start by examining how the temperature evolves in a flat radiation dominated Universe.
   (a) (*5 points*) Show that in a flat radiation dominated Universe the temperature T evolves as $T(t) = At^{-1/2}$. Express A in terms of physical quantities.
   (b) (*3 points*) What was the temperature five seconds after the Big Bang?
   (c) (*5 points*) Derive the expression for the particle horizon at time t, i.e. the proper distance at time t to the edge of the volume containing all particles that have been in causal contact with the observer, for a flat radiation dominated Universe.
   (d) (*8 points*) Assuming that one monopole forms in each horizon volume at the time when the temperature is T_M, use your expression for $T(t)$ you derived in question (a) to derive the number density of monopoles that are formed at this time. Express your result in terms of physical quantities.
   (e) (*5 points*) Recall that the photon number density at temperature T is

$$n_\gamma = \frac{2.4}{\pi^2} \left(\frac{k_B T}{\hbar c} \right)^3$$

   Calculate the ratio of the number density of monopoles to the photon number density at the time when the monopoles form.
   (f) (*5 points*) If this ratio of number densities does not evolve with time once the monopoles have formed, calculate the present day mass density of monopoles. Compare this with the present day critical density. Why is this a problem?

5. Consider a universe that only contains matter and a cosmological constant. In the previous problem set we derived that in this case the Friedmann equation is given by

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho(a) + \frac{\Lambda c^2}{3} - \frac{k}{a^2},$$

where a is the scale factor, $\rho(a)$ corresponds to the matter density, Λ is the value of the cosmological constant, and k is the curvature.

(a) (*3 points*) Use the definition of the density parameter to show that at any epoch we can write the matter density as

$$\frac{H^2(z)\Omega_m(z)}{(1+z)^3} = \Omega_{m,0}H_0^2 = \text{constant}.$$

(b) (*3 points*) Show that for spatially flat matter-dominated cosmologies with a cosmological constant the Friedmann equation can be written as

$$H^2(z) = H_0^2[1 - \Omega_{m,0} + \Omega_{m,0}(1+z)^3],$$

where H_0 is the present day value of the Hubble parameter and $\Omega_{m,0}$ the present day value of the density parameter $\Omega_m(z)$.

(c) (*3 points*) Show that in a spatially flat matter-dominated cosmology the density parameter evolves as

$$\Omega_m(z) = \Omega_{m,0}\frac{(1+z)^3}{1 - \Omega_{m,0} + (1+z)^3\Omega_{m,0}}.$$

(d) (*3 points*) Show that for this cosmology the comoving distance to an object with redshift z is given by

$$r_0 = \frac{c}{H_0}\int_0^z \frac{dz}{[1 - \Omega_{m,0} + \Omega_{m,0}(1+z)^3]^{1/2}}$$

(e) (*3 points*) For the special case of $\Omega_{m,0} = 1$ derive the equation for the angular diameter distance as a function of redshift.

(f) (*3 points*) Show that the Hubble parameter at decoupling ($z_{dec} \sim 1000$) can be expressed as

$$\frac{H(z_{dec})}{H_0} \simeq \sqrt{\Omega_{m,0}}(1 + z_{dec})^{3/2}.$$

As the only important characteristic scale in the young Universe, the Hubble length c/H gives the characteristic scale of the first peak in the microwave background. In a spatially-flat cosmology with a cosmological constant, the present radial coordinate distance to an object with $z \gg 1$ is given approximately by

$$r_0 \approx \frac{2c}{H_0\sqrt{\Omega_{m,0}}}$$

(g) (*3 points*) Show that the angle subtended by the Hubble length at decoupling is approximately independent of $\Omega_{m,0}$ in spatially-flat cosmologies, and compute its value in degrees.

6. No observational probe can constrain all cosmological parameters by itself with very high precision. Instead measurements from different methods are combined. This can be particularly effective if we consider observations of *distances* that are sensitive to different redshifts.

   (a) (*5 points*) Consider observations of type Ia supernovae at low redshift. Show that the joint constraints on $\Omega_{m,0}$ and $\Omega_{\Lambda,0}$ from such data are positively correlated (i.e. a larger value of the matter density also implies a larger cosmological constant).

   (b) (*5 points*) CMB measurements of the baryon acoustic oscillations probe very high redshifts. Show that in this case, negatively correlated joint constraints on $\Omega_{m,0}$ and $\Omega_{\Lambda,0}$ are expected.

   Combining these measurements dramatically improves the precision with which the basic cosmological parameters are determined, suggesting a flat geometry for our Universe. In this case the Friedmann equation can be used to obtain

$$\frac{1}{a}\frac{da}{dt} = H_0\sqrt{\frac{\Omega_{m,0}}{a^3} + 1 - \Omega_{m,0}}.$$

   (c) (*5 points*) Solve this differential equation to show that

$$t = \frac{2}{3H_0\sqrt{1-\Omega_{m,0}}}\operatorname{arcsinh}\left[\sqrt{\frac{1-\Omega_{m,0}}{\Omega_{m,0}}}a^{3/2}\right]$$