ORIGINS AND EVOLUTION OF THE UNIVERSE - PROBLEM SET #1

DUE WEDNESDAY OCTOBER 17, 2018

This is the first problem set for this course. The solutions to the problems in this set will be discussed on Wednesday October 17 from 13:30 in HL414. Your solutions will be graded if they are received at or before that time. There are six exercises in this set.

1. In class we derived the Friedmann equation by considering the Newtonian force exerted on a test mass by a sphere of mass M at a distance r. This derivation can be extended by adding a cosmological constant. This is done by adding an additional force. To ensure an isotropic and homogeneous universe, this force $\vec{F}_\Lambda$ has to be proportional to $\vec{r}$. For a test particle of unit mass, the convention is to express the force as

$$\vec{F}_\Lambda = \frac{\Lambda}{3}\vec{r}$$

(a) (5 points) Derive expressions for $(\dot{a}/a)^2$ and $\ddot{a}/a$, i.e., the Friedmann equation and the acceleration equation for this Newtonian cosmology with pressureless matter.

(b) (7 points) Show that these equations permit a static universe, which is why Einstein introduced the cosmological constant. How is Λ related to the density? What condition must the curvature satisfy?

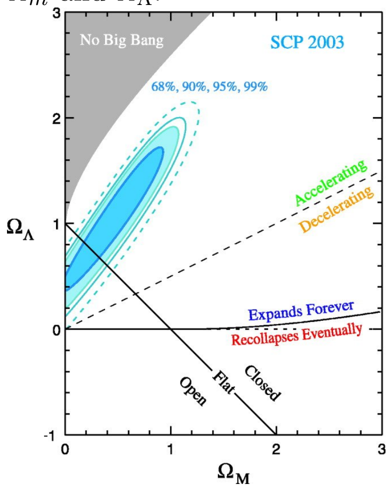
(c) (8 points) This universe is only static if $\dot{a} = 0$ to begin with, but is nonetheless unstable. This can be seen if we consider a small perturbation $\epsilon(t)$ to the scale factor: $a(t) = a_0[1 + \epsilon(t)]$, which results in a differential equation for $\epsilon(t)$. Derive this expression and find the solution for $\epsilon(t)$. Comment on the result.

2. Consider a flat universe in which the dark energy, cold pressureless matter and radiation contribute 70.24, 29.75 and 0.01 percent of the total energy density respectively.
(a) (5 points) At what redshifts is the energy density of matter equal to the energy density of radiation?

(b) (5 points) At what redshift is the contribution of matter to the energy density equal to that of dark energy in this universe?

(c) (5 points) Consider a universe with the same initial conditions as ours, but in which the matter particles have remained relativistic. How does the temperature of the CMB compare to that in our Universe when it is as old (no calculation is necessary, but motivate your answer)?

3. Here is a standard figure showing the implications of different parameter combinations Ω_m and Ω_Λ.



(a) (10 points) When the parameters $\Omega_\Lambda > 1.0$ and Ω_m is small, the above figure says "No Big Bang..." Why is this? Compute Ω_Λ for the boundary region at $\Omega_m = 0$, 0.5, 1.5, and 4

4. Consider a universe with only matter and curvature.
(a) (5 points) Use the Friedmann equation to show that expansion can only stop if $\Omega_{m,0} > 1$; what is the value of a_{max}, the scale factor at maximum expansion?

(b) (5 points) Using the substitution $y = a/a_{max}$ and $d\tau = a_{max}^{-1} H_0 \sqrt{\Omega_{m,0} - 1}dt$ show that

$$\frac{\sqrt{y}}{\sqrt{1-y}}dy = d\tau$$

(c) (10 points) Show that this results in an parametric expression for the time $t(\theta)$ where $\theta = 2\arcsin(\sqrt{y})$ given by

$$t(\theta) = \frac{1}{2H_0}\frac{\Omega_{m,0}}{(\Omega_{m,0}-1)^{3/2}}[\theta - \sin(\theta)]$$

(d) (5 points) Show that the scale factor is described by

$$a(\theta) = \frac{a_{max}}{2}[1 - \cos(\theta)]$$

(e) (5 points) How old is this universe when the Big Crunch occurs?

5. Hubble's law occurs naturally in a Big Bang model for the Universe, where the density decreases and the distances between galaxies increase as the Universe expands. This model

is not in agreement with the perfect cosmological principle, in which there are no privileged moments in time. The Big Bang itself is clearly a special moment. For this reason Bondi, Gold and Hoyle proposed the Steady State model in the 1940s. Interestingly, this model also gives rise to the Hubble law. In the Steady State model, the global properties of the Universe, such as H_0 and $\rho_m = \rho_0$, remain constant.
(a) (5 points) Show that the distance $r(t)$ increases with time t as $r(t) \propto e^{H_0 t}$

(b) (5 points) How does this model avoid the issues with the Big Bang?

(c) (5 points) For the Universe to remain in a steady state, the density of a volume V must remain constant, which means that matter must be created continuously at a rate $\dot{M}_{ss}$. Express $\dot{M}_{ss}$ in terms of H_0, ρ_0, and V. Calculate the rate with which matter would have to be created in our Universe, where $H_0 = 71$ km/s/Mpc and $\rho_0 = 3.0 \times 10^{-27}$ kg/m^3. Express your result in hydrogen atoms per cubic kilometer per year.

6. So far we have considered the redshift of an object to be fixed in time, but since it is a ratio of the scale factors now and at emission, the redshift will change. In this problem we consider a universe filled with material with a specific equation of state parameter w.
(a) (5 points) Differentiate the definition of redshift and use the Friedmann equation to show that

$$\frac{dz}{dt_0} = H_0(1+z) - H_0(1+z)^{3(1+w)/2}$$

(b) (5 points) If $w = 1/3$, what fractional precision in the observed wavelength would be required to detect cosmological deceleration in a decade?