Sterrewacht
Leiden

Origins & Evolution of the Universe

an introduction to cosmology — Fall 2018

Lecture 3: Intro GR, Space Time Metric, Distance

Rychard Bouwens

Layout of the Course

Sep 24: Introduction and Friedmann Equations
Oct 1: Fluid and Acceleration Equations
Oct 8: Introductory GR, Space Time Metric, Proper Distance
Oct 15: Redshift, Horizonts, Observable Distances, Parameter Constraints, Intro CMB
Oct 17: Problem Class #1
Oct 22: Observable Distances, Parameter Constraints
Oct 29: Thermal History, Early Universe
Nov 5: Early Universe, Inflation
Nov 12: Inflation, Lepton Era
Nov 14: Problem Class #2
Nov 19: Big Bang Nucelosynthesis, Recombination
Nov 26: Introduction to Structure Formation
Dec 3: Cosmic Microwave Background Radiation (I)
Dec 5: Problem Class #3
Dec 10: Cosmic Microwave Background Radiation (II)
Dec 21: Final Exam

You should have received problem set #1 by mail last week

Due by Wednesday 13:30 October 17, 2018

Review Last Week

Friedmann equation

Here we have an equation we can use to calculate how the scale factor *a* of the universe evolves with time!

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho - \frac{kc^2}{a^2}$$

Three cases:

k<0: expansion never stops, as $a \to \infty$, $da/dt > 0$

k=0: assymptotically slow expansion, as $a \to \infty$, $da/dt \to 0$

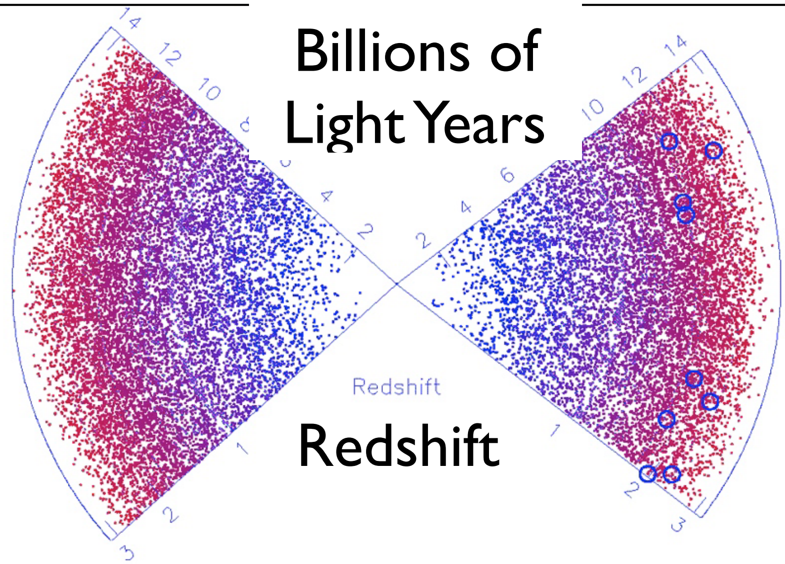
k>0: expansion stops and recollapses, as $a \to \infty$, $da/dt < 0$

with $a_{max} = (3kc^2/8\pi G\rho_0)^{1/2}$

Evidence for Homogeneity

One of the major assumptions we made in constructing our cosmological model is that of homogeneity and isotropy. Fortunately, the distribution of galaxies on the sky and especially the cosmic microwave background provide compelling evidence to support such assumptions!

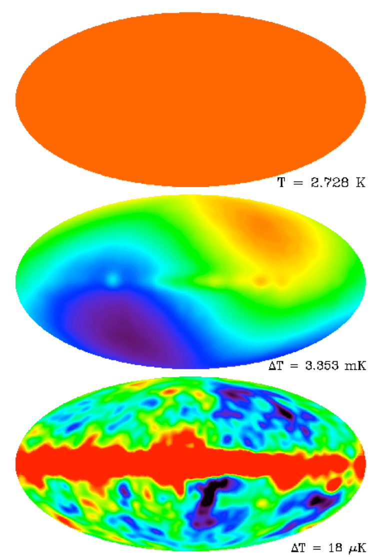
Galaxy Distribution
2dF Quasar Survey



More distant objects are more homogeneous. These results also support **isotropy**.

Cosmic Microwave Background

The CMB seen by COBE



The CMB is extremely uniform

The main feature is the dipole, due to our peculiar motion: the solar system is moving at 365 km/s w.r.t. the CMB

It also shows very small temperature fluctuations on the order of 1 part in 100,000 across the sky. Note the contamination by our galaxy.

Friedmann equation

How does the scale factor (size of universe) evolve (in the simplest case that k=0)?

$$\Rightarrow a = (t/t_0)^{2/3}$$

This case gives us the critical density:

$$\Rightarrow \rho_{crit} = (3H_0^2/8\pi G)$$

How can we recast the first Friedmann equation?

$$H^2(a) = H_0^2(\Omega_{matter,0} / a^3 + \Omega_{rad,0} / a^4 + \Omega_{\Lambda,0} + \Omega_{k,0}/a^2)$$

How does the density evolve?

Apply first law of thermodynamics to expanding universe:

$$dE + PdV = dQ$$

This results in the fluid equation:

$$d\rho/dt + 3((da/dt)/a)(\rho+P/c^2) = 0$$

To solve, we need to know equation of state: $P(\rho)$

Want to write as $P = w\varepsilon$

Find $\varepsilon_w = \varepsilon_{w,0}\, a^{-3(1+w)}$

For matter: $\Rightarrow w = 0$

For highly relativistic particles: $\Rightarrow w = 1/3$

We can combine the first of Friedmann's equations with the fluid equation to derive a formula for the acceleration of the scale factor, i.e., the second of Friedmann's equations:

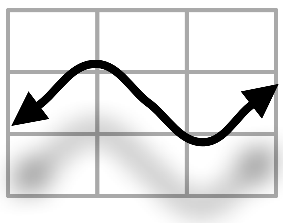
$$d^2a/dt^2 = -4\pi G/3\ (\rho + 3P/c^2)$$

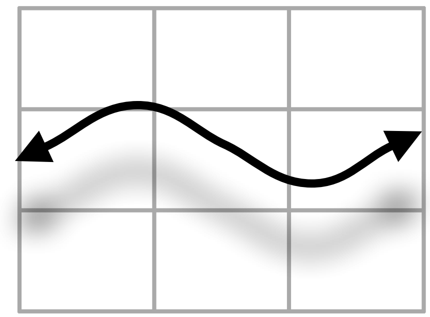
Impact of Expanding Universe on Properties of Light

We use the correlation between velocity and distance, i.e., Hubble's law, to infer an expanding universe

The velocity is inferred from the Doppler shifting or redshifting of the light…

This leads to a picture where photons are stretched according to the expansion of the grid points of the universe





$$\frac{\lambda_0}{\lambda_{emitted}} = \frac{\mathbf{a}\left(t_0\right)}{\mathbf{a}\left(t_{emitted}\right)}$$

Note 0 subscript indicates current date universe...

The expansion factor is the same for light, as for the universe itself

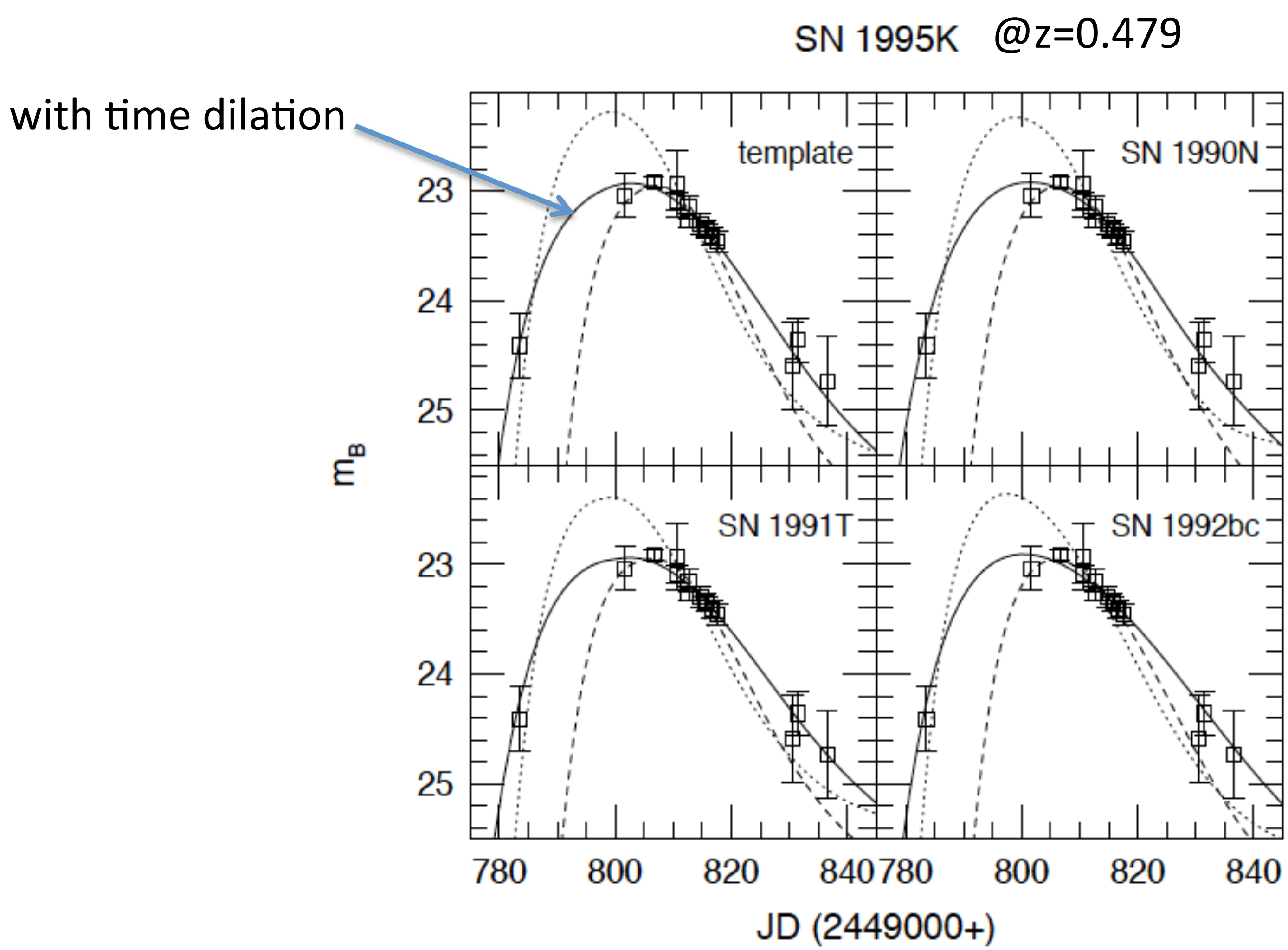
Many years ago, it was suggested that the redshifting of emitted light from distant source could also occur if light becomes “tired”

This would also result in more distant sources being redshifted.

But tired light theory would not predict that distant events be time delayed…

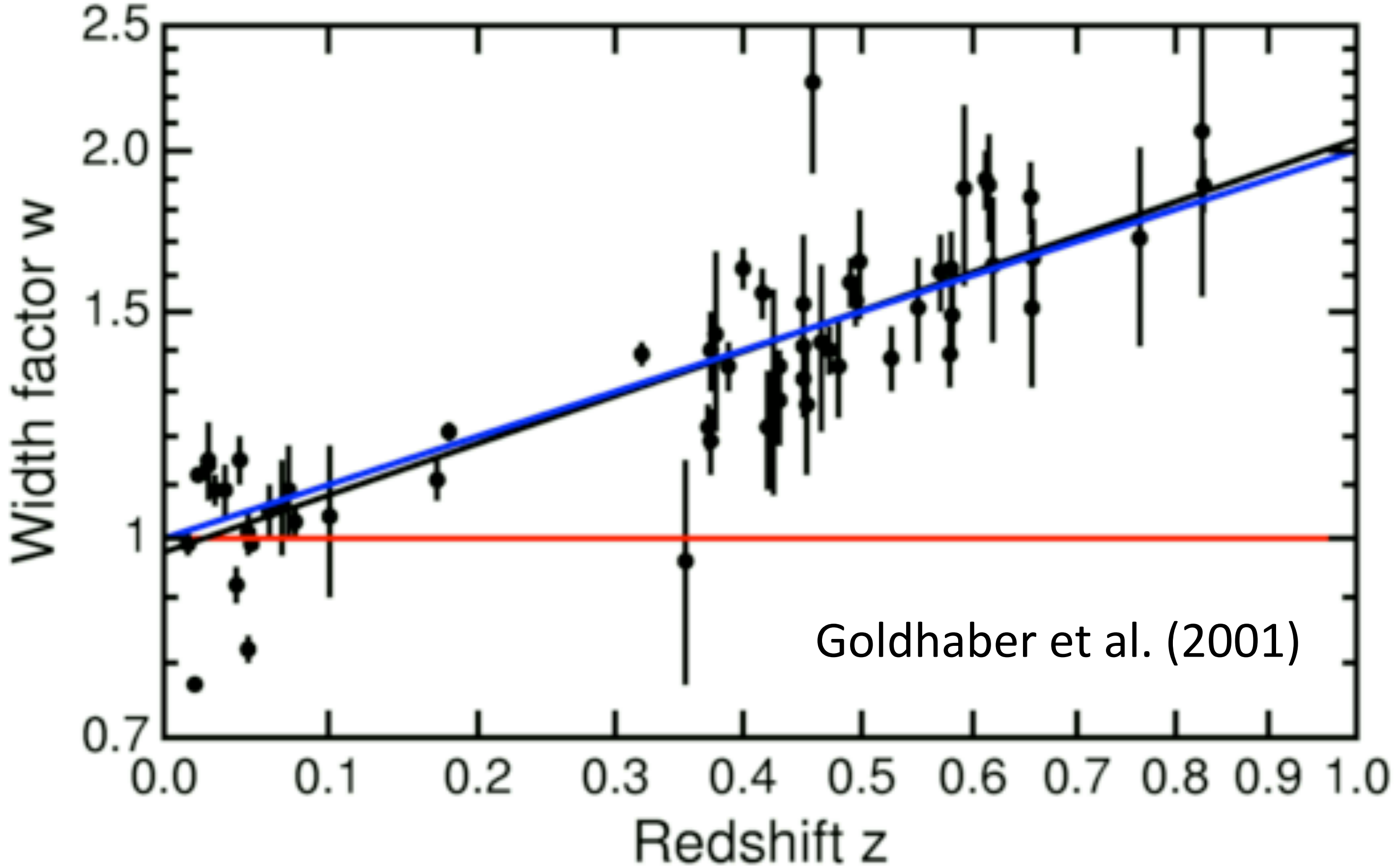
An expanding universe theory however predicts time delays as later events need to cover more distance (due to the extent to which the universe has expanded between events)

What is observed for the period of distant supernovae? Does the physical process take longer? YES!



Leibundgut et al. (1996)

Interpretation of redshift



Supernovae at greater redshifts are seen to take longer to decay: it scales linearly with the redshift: the redshifts are true reflections of Doppler shift.

ASIDE: Energy Density in CMB Radiation

What is Ω_{rad} from the CMB?

It can be simply calculated. We know the temperature of the cosmic microwave background CMB using the energy density formula for black body radiation.

$$\varepsilon_{rad} = \alpha T_{rad}^4 \qquad\qquad \alpha = \pi^2 k_B^4 / (15\hbar^3 c^3)$$

We know the CMB retains a blackbody spectrum (since it originally had a black body due to the equilibriation that happened during Big Bang and due to the fact that the expansion of the universe is adiabatic).

The energy density in radiation relative to the critical density of the universe is

$$\Omega_{rad} = \varepsilon_{rad} / c^2 / \rho_{crit}$$

Deriving an Expanding Universe using General Relativity

Introduction to Concepts Relevant to General Relativity

General Relativity based on the observation that the inertial and gravitational masses are the same

Gravitational Force $F = m_g\, GM/r^2$

the same why?

Impact of Force on Mass $F = m_i\, a$

Assume instead that mass curves space time such that test particles naturally accelerate in a way independent of the mass of the test particle.

In GR, particles move along geodesics (shortest paths)

Space time described by metric $g_{\mu\nu}$ which gives distance ds between events

$\mathbf{x} = (t,x,y,z)$ and $\mathbf{x} + \mathbf{dx} = (t + dt, x + dx, y + dy, z + dz)$

Metric and Coordinate System

The geometric properties of the surface can be obtained by considering the distance between a close of infinitesimally close points:

$$dl^2 = \Sigma_{i,j}\, g_{ij}(x)\; dx^i\, dx^j$$

metric

In cartesian coordinates (x,y) $\Rightarrow g_{ij} = \delta_{ij}$ and $dl^2 = dx^2 + dy^2$

Space Time

Space time is described by the distance between two events:

$$ds^2 = c^2\, dt^2 - dx^2 - dy^2 - dz^2$$

$$= \eta_{\mu\nu}\, x^\mu\, x^\nu = c^2 dt^2 - \delta_{ij}\, x^i\, x^j$$

where $(x^0, x^1, x^2, x^3) = (ct, x, y, z)$ and $\eta_{\mu\nu} = \mathrm{diag}(1,-1,-1,-1)$

For a general space time: $ds^2 = g_{\mu\nu}\, dx^\mu\, dx^\nu$

Since ds is invariant under general coordinate transformations, the metric must transform as follows:

$$g_{\mu\nu\,(x)} \rightarrow g'_{\mu\nu\,(x)} = (\partial x^\alpha/\partial x'^\mu)\,(\partial x^\beta/\partial x'^\nu)\; g_{\alpha\beta\,(x)}$$

The inverse four-metric $g^{\mu\nu}$ is the inverse of $g_{\mu\nu}$ $\Rightarrow g_{\mu\alpha}\, g^{\alpha\nu} = \delta_\mu{}^\nu$

According to the principle of general relativity, all reference frames are equivalent and a physical law should have the same form under a general coordinate transformation (the general covariance)

Curvature of Space Time

For reference — not in lecture

In general relativity, the curvature of space time is important: particles move such that

$\delta \int_{path} ds = 0$ the integral is stationary

In the reference frame coming with the particle (where according to the space-time must be locally Minkowski with metric $\eta_{\mu\nu}$), the motion of a particle must be given by

$d^2\xi^\mu / ds^2 = 0$

where ds/c is the proper time measure in the free fall frame

where ξ^μ is the space time coordinate of the particle

For a general reference frame with coordinate x^μ related to ξ^μ by $x^\mu(\xi)$, the metric $g_{\mu\nu}$ is related to $\eta_{\mu\nu}$ by

$$g_{\mu\nu} = \eta_{\alpha\beta} (\partial\xi^\alpha/\partial x^\mu) (\partial\xi^\beta/\partial x^\nu)$$

Curvature of Space Time

For reference — not in lecture

In the x-frame, the equation of motion becomes

$$d^2x^\mu / ds^2 = -\Gamma^\mu{}_{\alpha\beta}\,(dx^\alpha/ds)(dx^\beta/ds)$$

where $\Gamma^\mu{}_{\alpha\beta} = (\partial x^\mu/\partial \xi^\nu)(\partial^2 \xi^\nu/\partial x^\alpha \partial x^\beta)$ is the Christoffel symbol or affine connection

$$\Gamma^\mu{}_{\alpha\beta} = (1/2)g^{\mu\sigma}(\partial_\beta g_{\sigma\alpha} + \partial_\alpha g_{\sigma\beta} - \partial_\sigma g_{\alpha\beta})$$

Thus in the x-frame there is a force exerting on the free-fall of a particle, i.e., gravity.

In the perspective of GR, it is because the particle is moving in curved space.

Four Momentum and Energy-Momentum Tensor

For a particle of rest mass m, we can define a four momentum as
$p^\mu = mU^\mu$, with $U^\mu = c\,dx^\mu/ds$

In terms of the individual components this is equal to the following:

$(E/c, p_x, p_y, p_z)$ where the energy E is $(m^2c^4+p^2c^2)^{1/2}$

In Newtonian and special relativity, the conservation of mass, energy, and momentum + equivalence of mass and energy leads to

$\partial T_{\mu\nu}/\partial x^\mu = 0$ where $T_{\mu\nu}$ is the energy momentum tensor

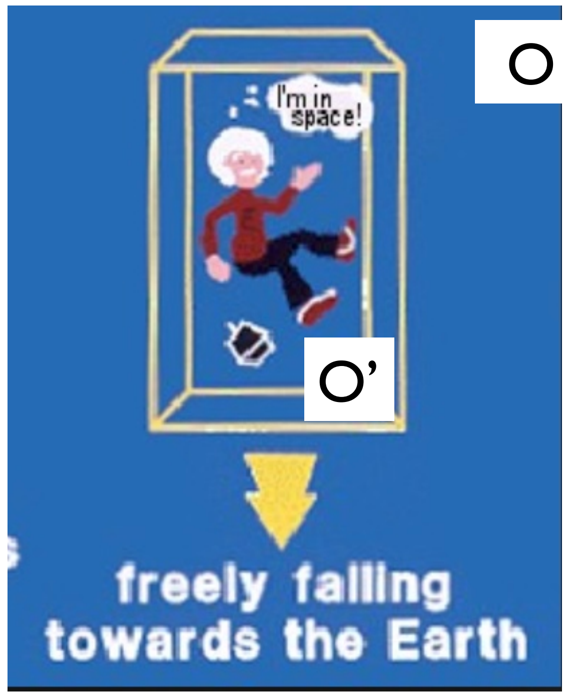
$T_{\mu\nu}$ describes the matter distribution for a perfect fluid (no viscosity, heat flow, or stress) with pressure P and energy density ρ

$T_{\mu\nu} = (P + \rho c^2)U_\mu U_\nu - Pg_{\mu\nu}$ as $U_\mu = g_{\mu\nu} U^\nu = g_{\mu\nu} (c\,dx^\nu/ds)$

$dx^\nu(s)$ is the worldline s of a fluid element

Let's examine Equivalence Principle using different reference frames, but do it in the Newtonian Limit

The Form of the space-time metric in the Newtonian limit of gravity will tell us how Newtonian gravity is interpreted in terms of geometric quantities:



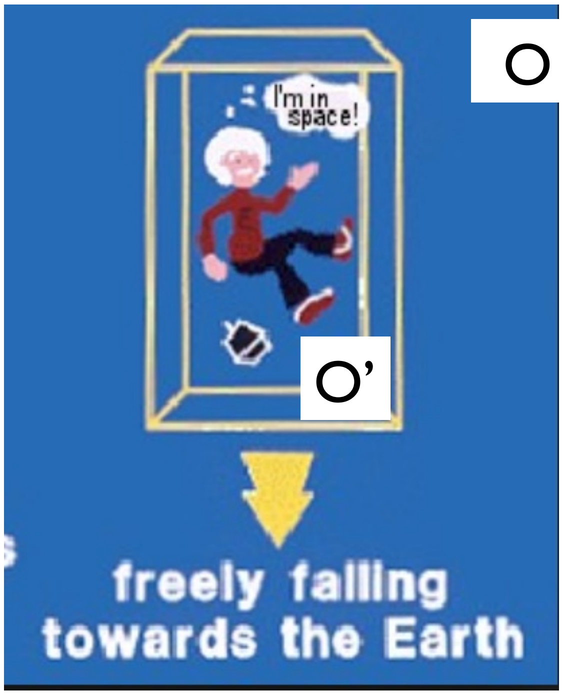
Consider a reference frame O' (free fall in a Newtonian potential Φ), i.e., frame of falling elevator)
Then the metric is Minkowski:

$$ds^2 = c^2\, dt'^2 - dx'^2$$

Consider another reference frame O (O' has the free-fall velocity relative to O of $v^2 = -2\,\Phi$ in the x-direction).

Let's examine Equivalence Principle using different reference frames, but do it in the Newtonian Limit

The Form of the space-time metric in the Newtonian limit of gravity will tell us how Newtonian gravity is interpreted in terms of geometric quantities:



O (not-accelerating)

Lorentz Transformation between them is

$dt' = (1+2\Phi/c^2)^{1/2}\, dt$ $\qquad$ $dx' = (1-2\Phi/c^2)^{1/2}\, dx$

Thus, the metric in terms of coordinates in the O reference frame (non-accelerating) is

$$ds^2 = c^2\,(1+2\Phi/c^2)dt^2 - (1-2\Phi/c^2)(dx^2+dy^2+dz^2)$$

$$\text{so } g_{00} = 1+2\Phi/c^2$$

Newtonian Limit → Einstein's Field Equation

Start with Gravitation in Newtonian Form:

Poisson's equation
$(\nabla^2\Phi=4\pi G\rho)$

$$(\nabla^2\Phi=4\pi G\rho)$$

Write the left hand side in terms of derivations of a curved space time metric

Note:

$\nabla^2\Phi = (1/2)\nabla^2(2\Phi)$

$= (c^2/2)\nabla^2(1+2\Phi/c^2)$

$= (c^2/2)\nabla^2 g_{00}$

Write the right hand side using energy momentum tensor $T_{\mu\nu}$

In the Newtonian limit
$T_{00} = \rho c^2$

$$(c^2/2)\nabla^2(g_{00}) = 4\pi G(T_{00} / c^2)$$

$$\nabla^2(g_{00}) = 8\pi G(T_{00} / c^4)$$

Newtonian Limit → Einstein's Field Equation

$$\nabla^2 (g_{00}) = 8\pi G(T_{00} / c^4)$$

From this hypothesize the Newtonian Limit can be extended to a full tensor form of this equation and the left hand side can be represented by tensor curvature term. Many potential possibilities, but most don't work:

Einstein proposed the following tensor (called Einstein's tensor):

$$G_{\mu\nu} = R_{\mu\nu} - g_{\mu\nu} R$$

Ricci tensor

Ricci scalar

where the Rici tensor and scalar include many different spatial derivatives of various Christoffel tensors and tensor products of various Chrisoffel tensors.

Plugging in Einstein's tensor on the left hand side of the above equation, we can translate the above tensor equation into Einstein's general field equation:

$$G_{\mu\nu} = 8\pi G T_{\mu\nu} / c^4$$

Including A Cosmological Constant in Einstein's Field Equation

It is possible to write a modified set of field equations that are consistent with the conservation laws:

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = (8\pi G/c^4)T_{\mu\nu}$$

cosmological constant

This modification allows for a static universe

What is the physical meaning of Λ?

It is useful to move Λ to the right hand side: $T_{\mu\nu}^{vac} = (-c^4\Lambda/8\pi G)g_{\mu\nu}$

$$G_{\mu\nu} = (8\pi G/c^4)(T_{\mu\nu} + T_{\mu\nu}^{vac})$$

If we recall that $T_{\mu\nu} = (P + \rho c^2)U_\mu U_\nu - Pg_{\mu\nu}$, the Λ term can be included as an ideal fluid with

$$\rho = -P/c^2 = (-c^2\Lambda/8\pi G)$$

Note in the case of vacuum energy: $\rho c^2 + 3P < 0$

Pressure as a source of gravity: Newtonian Gravity in a relativistic fluid (where we cannot assume $P << \rho c^2$)

$$\nabla^2\Phi=4\pi G(\rho+3P/c^2)$$

Curved Space Time

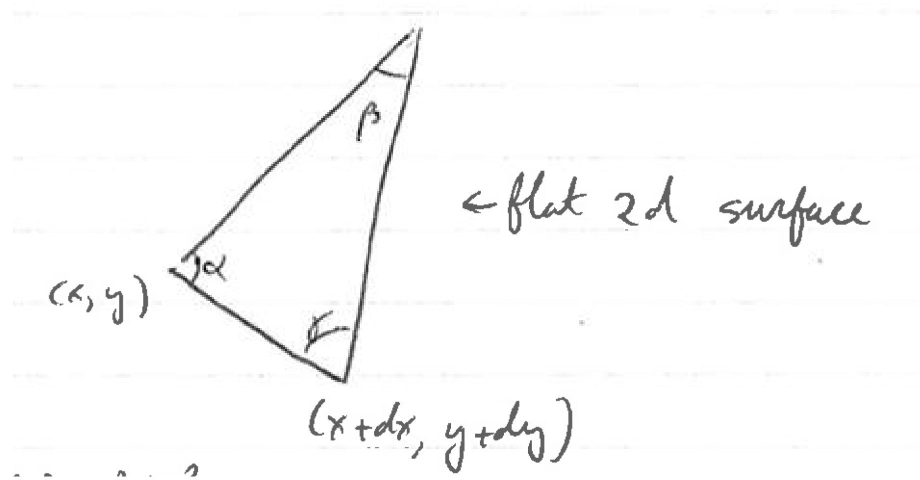
In general relativity, the metric is key, but which one describes the universe and obeys the cosmological principle?

Curvature must be the same everywhere1

Start with 2D spaces

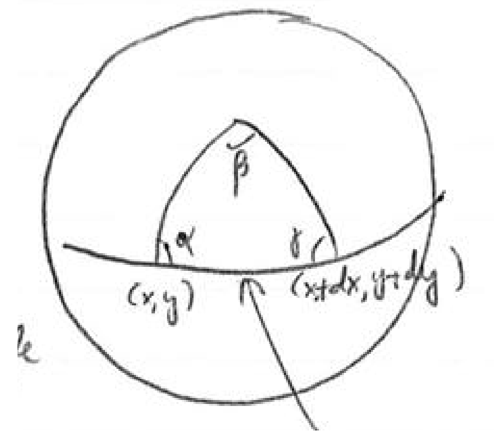
$\alpha + \beta + \Upsilon = \pi$ (in radians)

we know that $ds^2 = dx^2 + dy^2$



Curved Space Time

Now curved space - surface of sphere:



Area of Triangle

$$\alpha + \beta + \Upsilon = \pi + A/R^2$$

Radius of Sphere

$\alpha + \beta + \Upsilon > \pi$: positively curved

Curvature is homogenous + isotropic

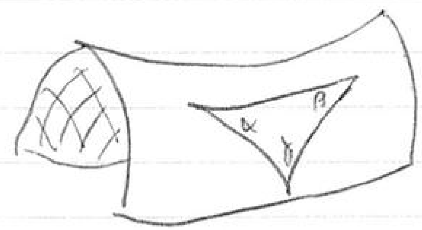
Note maximum distance between points of πR

Volume of $4\pi R^2$

Curved Space Time

Similarly we can define for negatively curved space

$$\alpha + \beta + \Upsilon = \pi - A/R^2$$



infinite area, no upper limit on distance

Infinite Volume + No Maximum Distances

$\alpha + \beta + \Upsilon < \pi$: negatively curved

General Curved Space Time:

Flat Space Time: $ds^2 = dx^2 + dy^2 + dz^2$

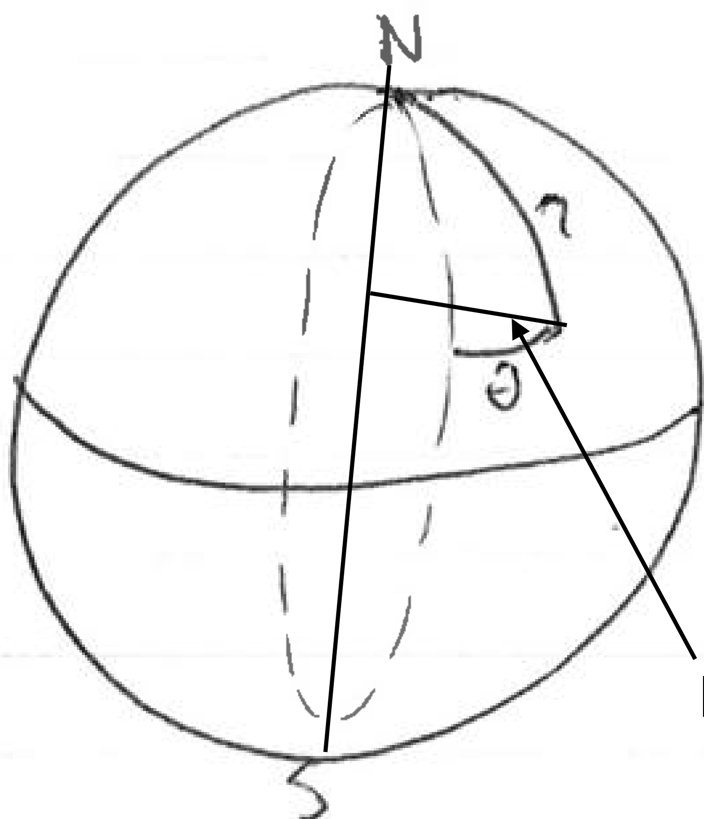
$$ds^2 = dr^2 + r^2(d\theta^2 + \sin^2\theta \, d\varphi^2)$$

General Curved Space Time:

Flat Space Time: $ds^2 = dx^2 + dy^2 + dz^2$

$$ds^2 = dr^2 + r^2(d\theta^2 + \sin^2\theta \, d\varphi^2)$$

Positively Curved: $ds^2 = dr^2 + R^2\sin^2(r/R) \, (d\theta^2 + \sin^2\theta \, d\varphi^2)$



Radius R

another dimension not visualizable in 3D

$R \sin(r/R) = S(r)$

2D manifold in 3D

General Curved Space Time:

Flat Space Time: $ds^2 = dx^2 + dy^2 + dz^2$

$$ds^2 = dr^2 + r^2(d\theta^2 + \sin^2\theta \; d\varphi^2)$$

Positively Curved: $ds^2 = dr^2 + R^2\sin^2(r/R) \; (d\theta^2 + \sin^2\theta \; d\varphi^2)$

Negatively Curved: $ds^2 = dr^2 + R^2 \sinh^2(r/R)(d\theta^2 + \sin^2\theta \; d\varphi^2)$

All these metrics have constant curvature:

$$ds^2 = dr^2 + S^2(r) \; (d\theta^2 + \sin^2\theta \; d\varphi^2)$$

where

$$S(r) = \begin{cases} R\sin(r/R), & \text{if } k = +1 \\ r, & \text{if } k = 0 \\ R\sinh(r/R), & \text{if } k = -1 \end{cases}$$

General Curved Space Time:

If we change coordinate system such that $x = S(r)$

$$
\text{i.e. } x = S(r) = \begin{cases} R\sin(r/R), & \text{if } k = +1 \\ r, & \text{if } k = 0 \\ R\sinh(r/R), & \text{if } k = -1 \end{cases}
$$

$$R \sin^{-1}(x/R) = r$$
$$dx/(1-x^2) = dr$$

$$R \sinh^{-1}(x/R) = r$$
$$dx/(1+x^2) = dr$$

Then,

$$ds^2 = dx^2/(1+kx^2/R^2) + x^2\,(d\theta^2 + \sin^2\theta\, d\varphi^2)$$

or using r instead of x:

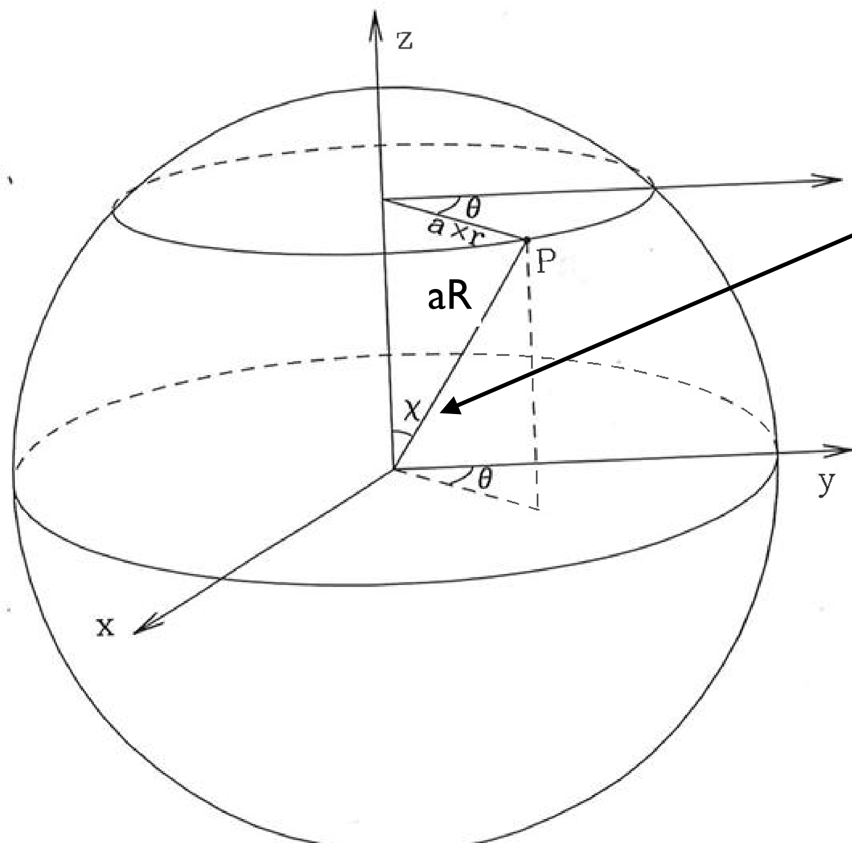
$$ds^2 = dr^2/(1+kr^2) + r^2\,(d\theta^2 + \sin^2\theta\, d\varphi^2)$$

General Curved Space Time:

Flat Space Time: $ds^2 = dx^2 + dy^2 + dz^2$

$$ds^2 = dr^2 + r^2(d\theta^2 + \sin^2\theta \, d\varphi^2)$$

Positively Curved: $ds^2 = dr^2 + R^2\sin^2(r/R)\,(d\theta^2 + \sin^2\theta \, d\varphi^2)$



$X = \sin^{-1}(r/R)$

General Curved Space Time:

This is the most general spatial metric with constant curvature. The only change we need to make is to allow space to expand:

$$ds^2 = -\,c^2dt^2 + a^2(t)\,[dr^2/(1+kr^2) + r^2\,(d\theta^2 + \sin^2\theta\,d\varphi^2)]$$

Proper Distance

Imagine we have a distant galaxy at (r, θ, φ)

How far away is a distant galaxy?

(important to specify time since nominal distance depends on when measurement is made since expanding universe)

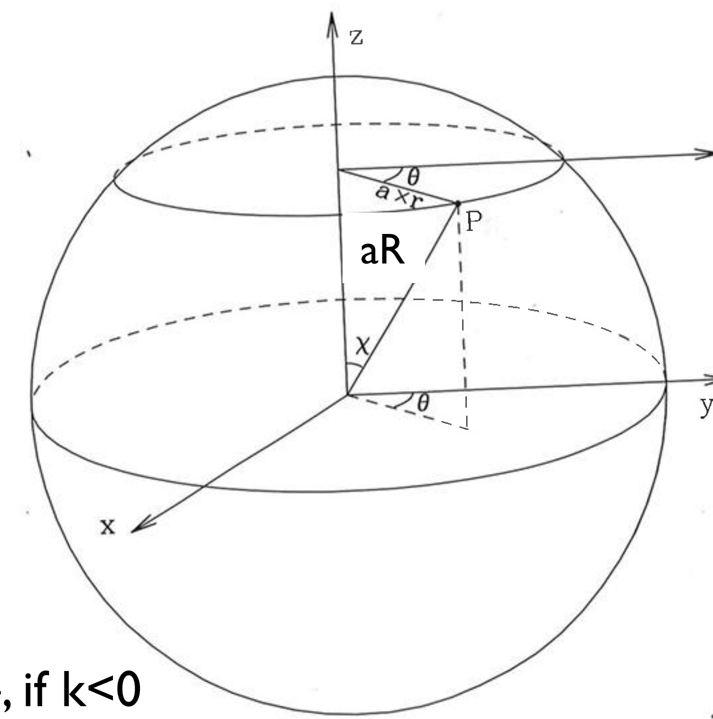
Take case that θ, φ are constant and only r is different

In the flat case (k=0),

Proper Distance: $d_p(t) = \int ds$

"Source-source distance measured based on current topology of the universe and ignoring travel time"

$$ds^2 = a^2(t)\,[dr^2/(1-kr^2) + r^2(d\theta^2 + \sin^2\theta d\varphi^2)]$$



In general case,

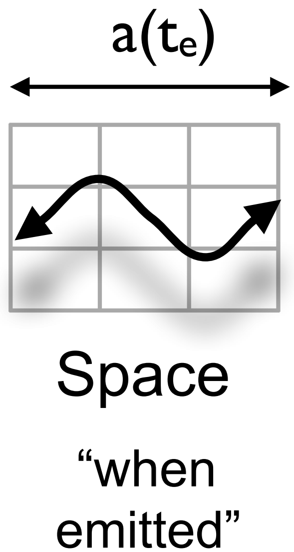
$$\text{Proper Distance: } d_p(t) = a(t)\underbrace{\begin{cases} \arcsin(k^{1/2}r)/k^{1/2}, & \text{if } k>0 \\ r, & \text{if } k=0 \\ \operatorname{arcsinh}((-k)^{1/2}r)/(-k)^{1/2}, & \text{if } k<0 \end{cases}}_{f_k(r)}$$

Not especially practical (since not measurable)!

Distances and Redshifts

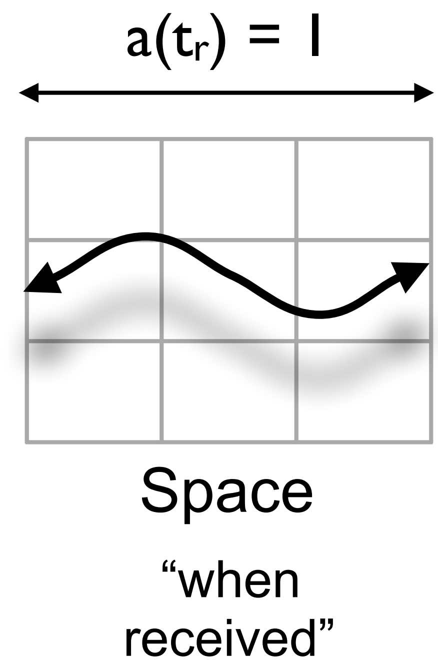
We cannot measure proper distances, but we can measure redshifts.

Redshift — which we denote as z — is directly connected to the scale factor of the universe.



$$\lambda_e (1+z) = \lambda_r$$

$$a(t_e) = 1/1+z$$



Redshift — which we denote as z — is directly connected to the scale factor of the universe.

Distances and Redshifts

Now let us to demonstrate how redshift relates to scale factor using the metric.

(Again take the case that θ, φ are constant and only r is different)

Since light travels on geodesics, ds = 0 and so

$$c\,dt / a(t) = dr / (1 - kr^2)^{0.5}$$

Consider the crest of a wave and consider it traveling from coordinate r = 0 to r_{obs}:

$$\int_0^{r_{obs}} dr / (1 - kr^2)^{0.5} = \int_{t_e}^{t_r} c\,dt / a(t)$$

It will make this trip from some time the t_e some later time t_r

Imagine some short time later… $t_e + dt_e$, $t_r + dt_r$ that the next wavefront moves between the same coordinates

$$\int_0^{r_{obs}} dr / (1 - kr^2)^{0.5} = \int_{t_e+dt_e}^{t_r+dt_r} c\,dt / a(t) \;\; = \text{above expression}$$

We can subtract $\int_{t_e+dt_e}^{t_r} c\,dt / a(t)$ from both

$$\Longrightarrow \int_{t_e}^{t_e+dt_e} c\,dt / a(t) = \int_{t_r}^{t_r+dt_r} c\,dt / a(t)$$

Distances and Redshifts

We can subtract $\int_{t_e+dt_e}^{t_r} c\ dt\ /\ a(t)$ from both $\Longrightarrow \int_{t_e}^{t_e+dt_e} c\ dt\ /\ a(t) = \int_{t_r}^{t_r+dt_r} c\ dt\ /\ a(t)$

$$\Longrightarrow c\ dt_e\ /\ a(t_e) = c\ dt_r\ /\ a(t_r)$$

As such, $dt \propto a(t)$ implies time dilation

Since the delta interval correspond to different crests of a wave,

$$\lambda \propto dt \propto a(t)$$

This implies

$$\lambda_r/\lambda_e = a(t_r)/a(t_e)$$

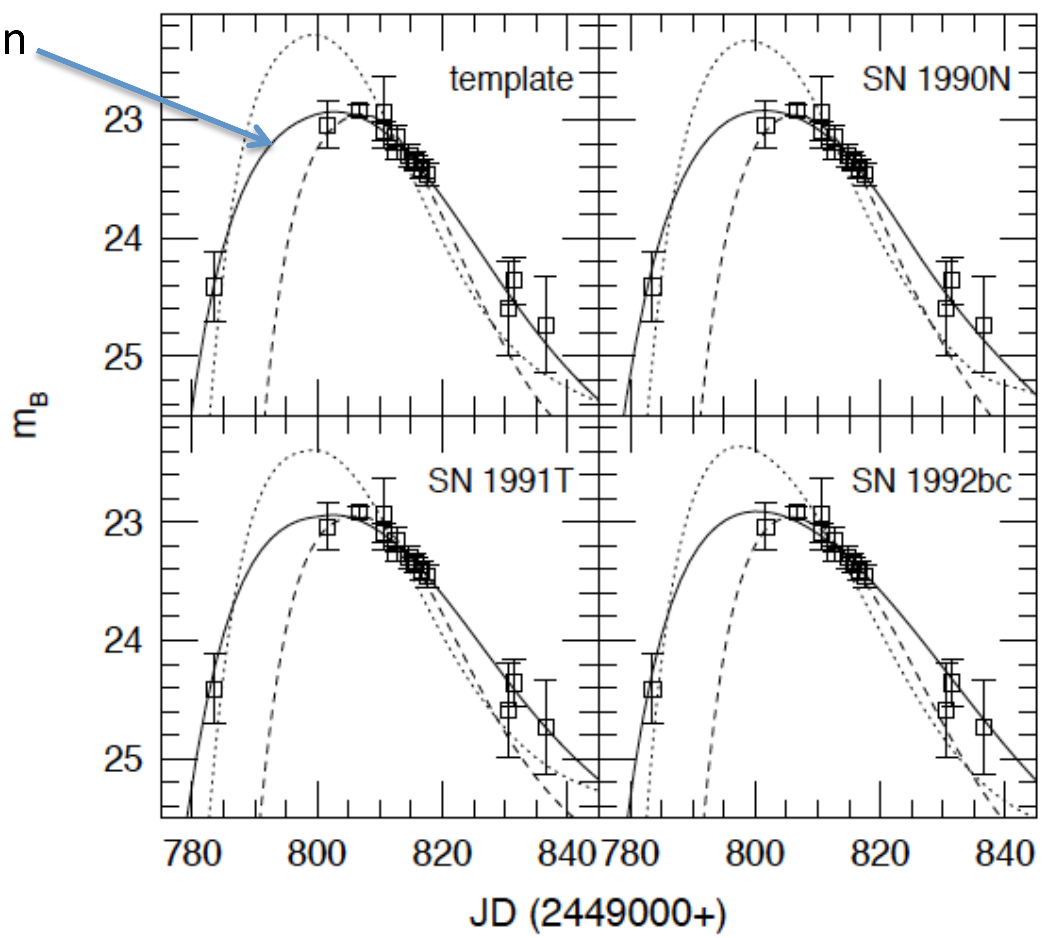
Since $a(t_r) = 1$ and $\lambda_r/\lambda_e = 1+z$,

$$a(t_e) = 1/1+z$$

Observation of time dilation

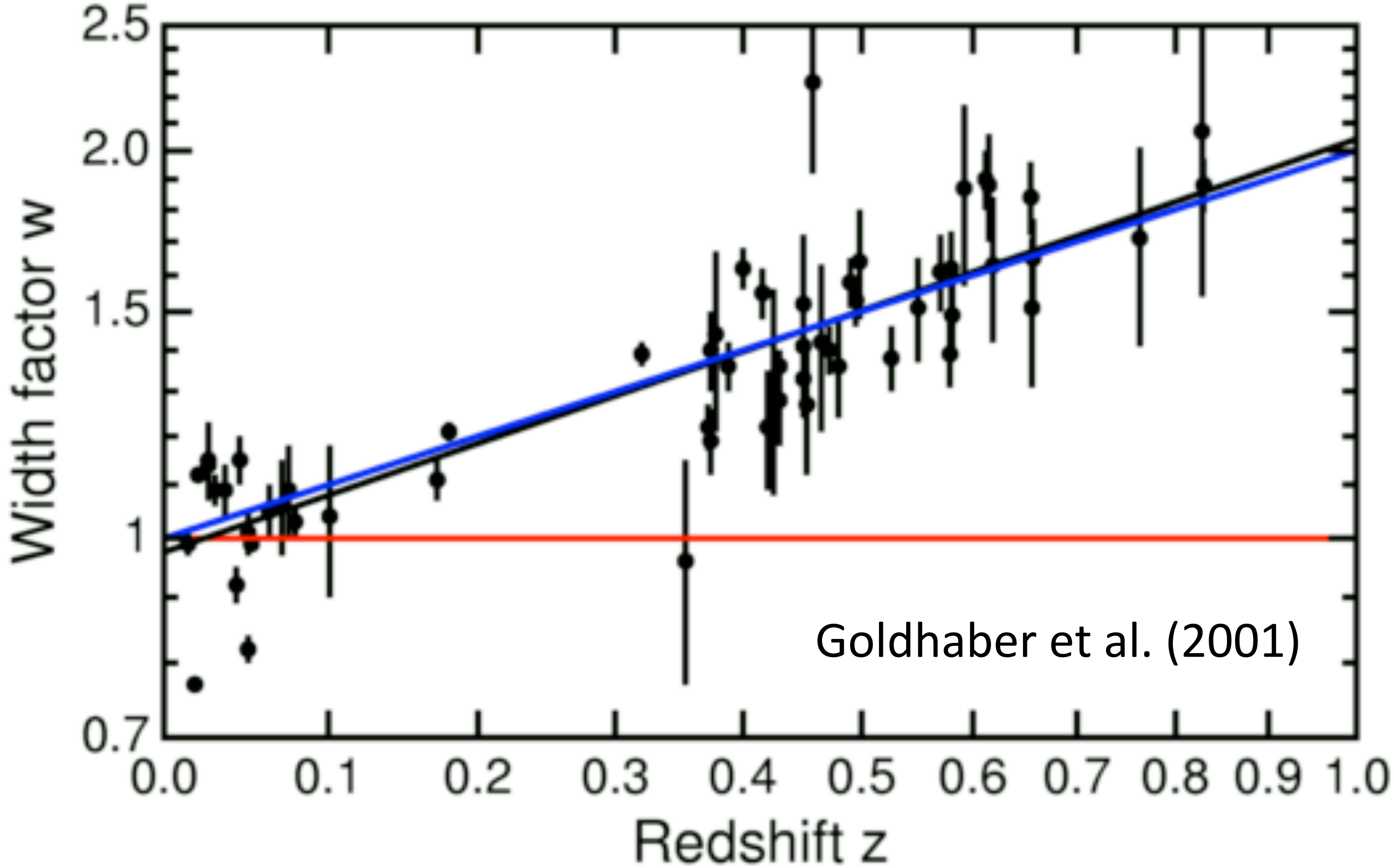
with time dilation

SN 1995K @z=0.479



Leibundgut et al. (1996)

Interpretation of redshift



Supernovae at greater redshifts are seen to take longer to decay: it scales linearly with the redshift: the redshifts are true reflections of Doppler shift.