Sterrewacht Leiden

Origins & Evolution of the Universe

an introduction to cosmology — Fall 2018

Lecture 2: Fluid and Acceleration Equations

Rychard Bouwens

Layout of the Course

Sep 24: Introduction and Friedmann Equations
Oct 1: Fluid and Acceleration Equations ←
Oct 8: Introductory GR, Space Time Metric, Proper Distance
Oct 15: Redshift, Horizonts, Observable Distances, Parameter Constraints, Intro CMB
Oct 17: Problem Class #1
Oct 22: Observable Distances, Parameter Constraints
Oct 29: Thermal History, Early Universe
Nov 5: Early Universe, Inflation
Nov 12: Inflation, Lepton Era
Nov 14: Problem Class #2
Nov 19: Big Bang Nucelosynthesis, Recombination
Nov 26: Introduction to Structure Formation
Dec 3: Cosmic Microwave Background Radiation (I)
Dec 5: Problem Class #3
Dec 10: Cosmic Microwave Background Radiation (II)
Dec 21: Final Exam

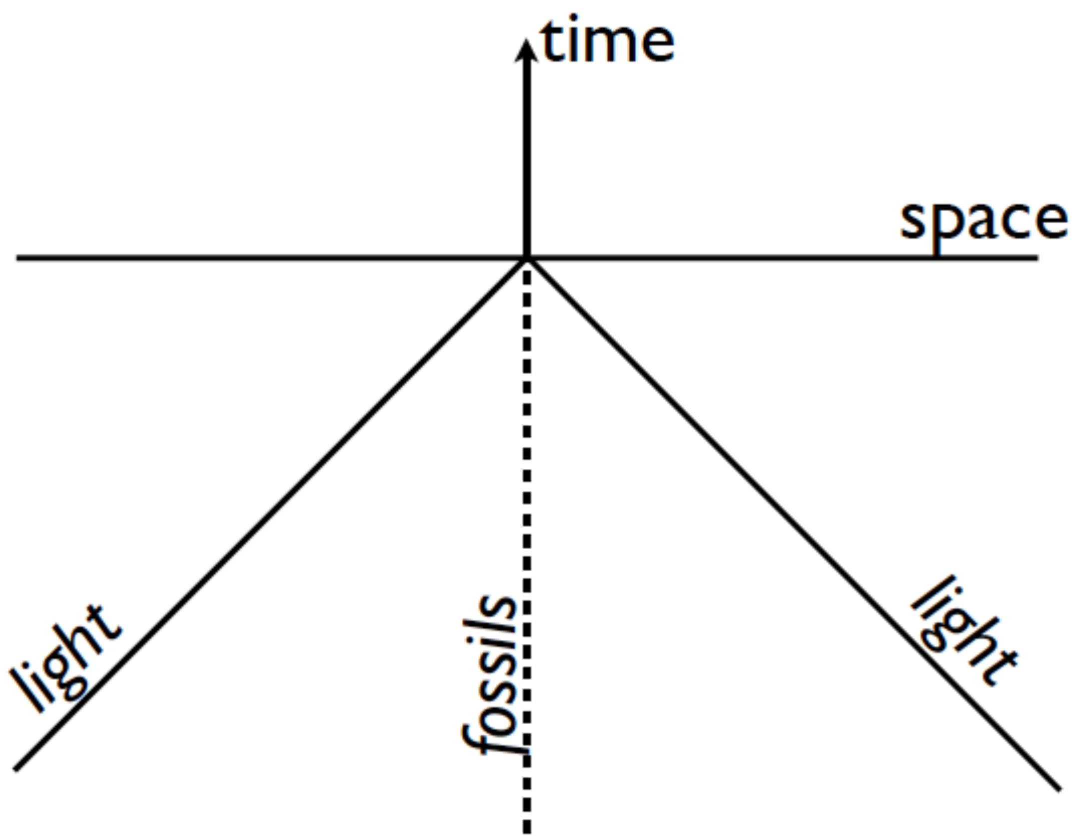
Expect to receive problem set #1 by mail in the next few days

Due by 13:30 on Wednesday, October 17

Review Last Week

Need for homogeneity

To be able to relate observations along different lines of sight we need to assume **homogeneity**: *without this assumption we cannot interpret observations*.

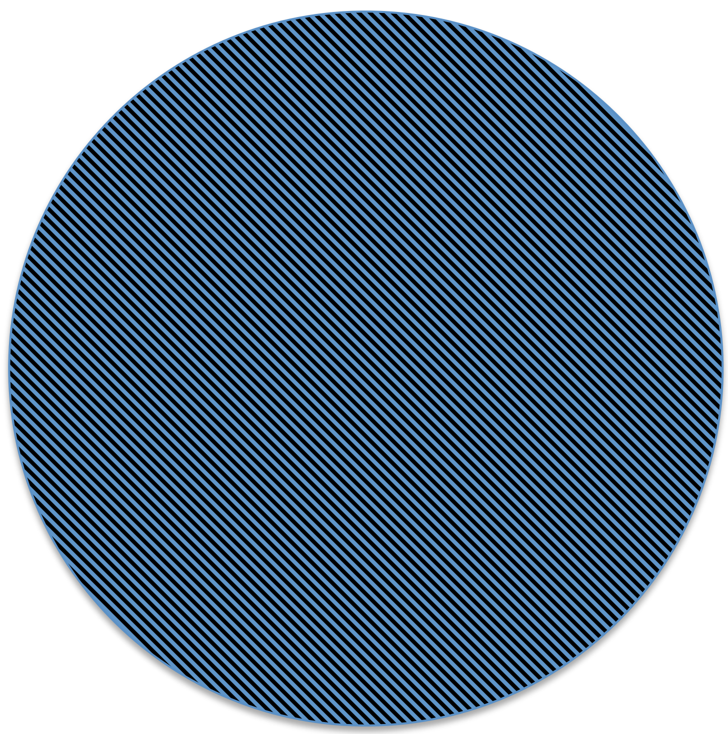


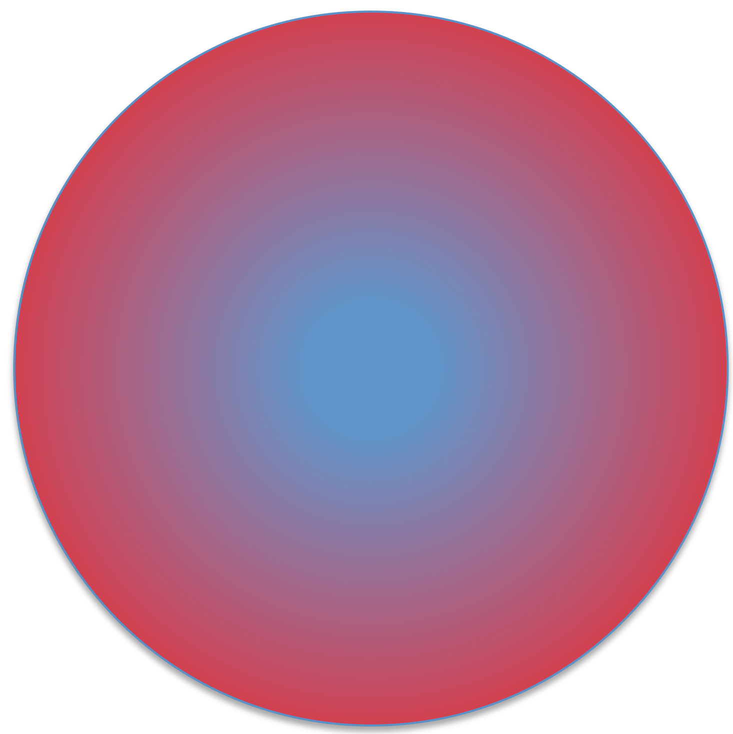
Homogeneity and Isotropy

In building a cosmological model, we assume homogeneity + isotropy

Homogeneity: the Universe looks the same at each point.
Isotropy: the Universe looks the same in all directions.

Homogeneity does not imply isotropy, and isotropy does not imply homogeneity.





Velocity-Distance diagram

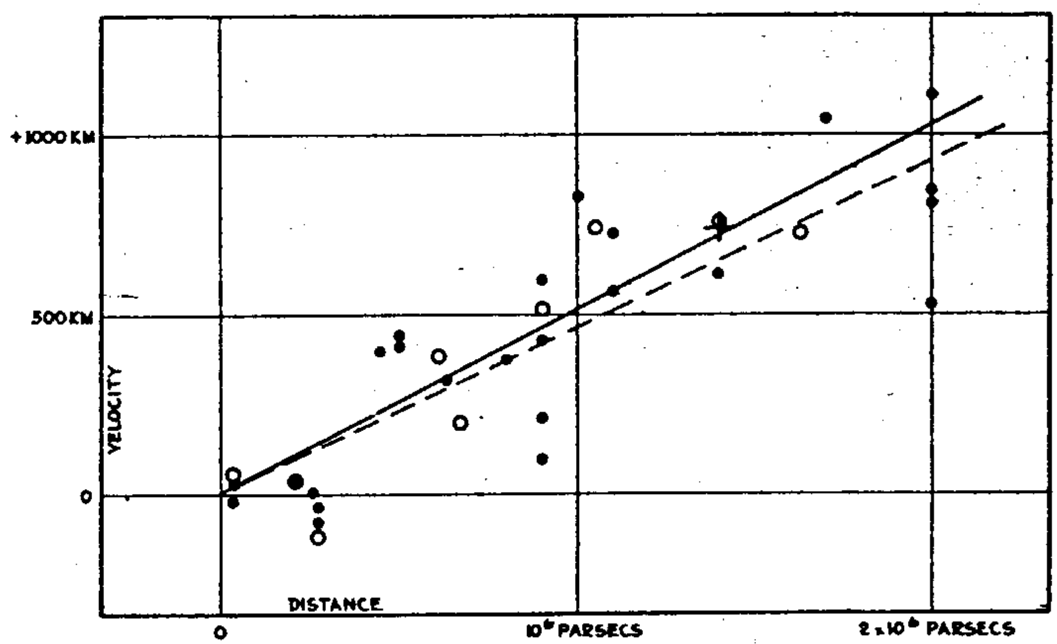
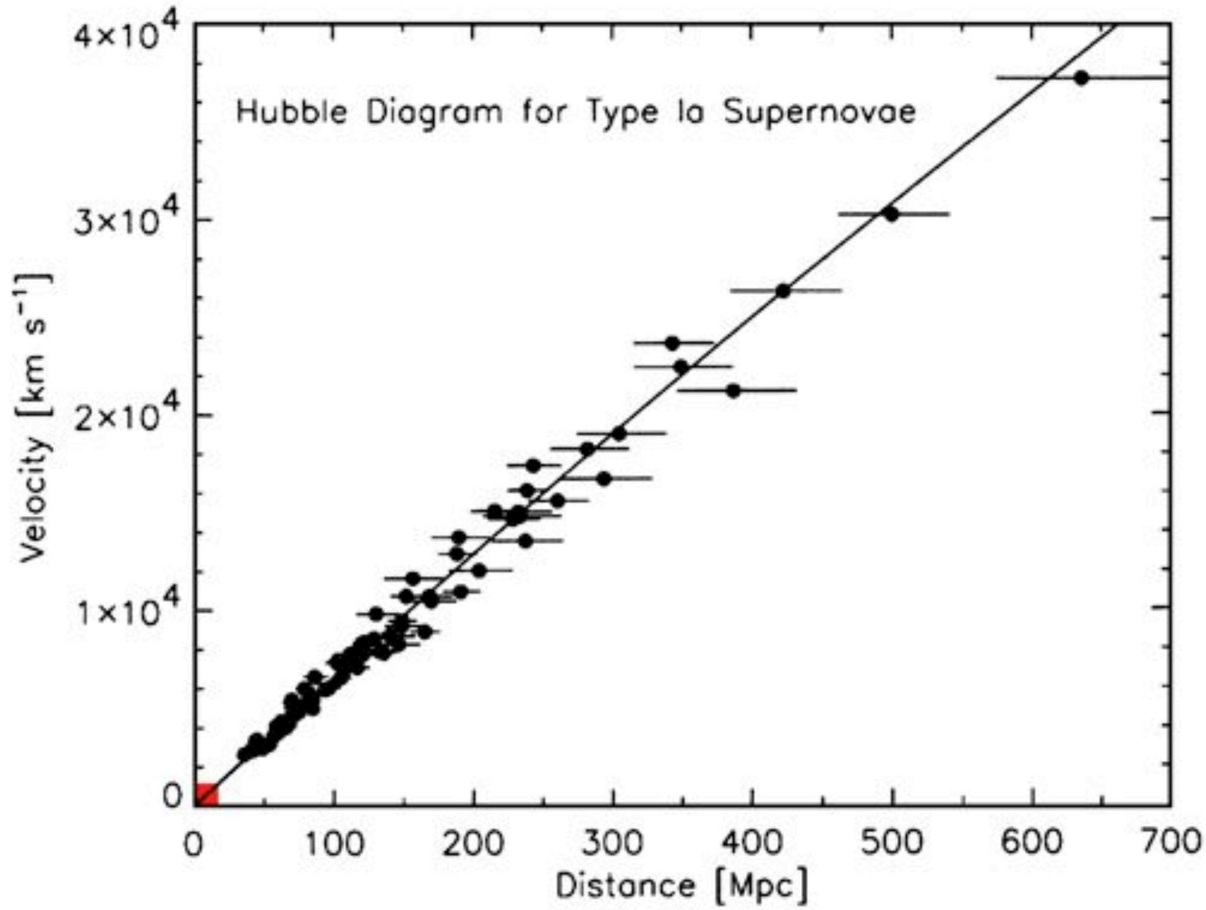


FIGURE 1

Hubble (1929): the recession velocity is proportional to the distance (after allowing for the Sun's motion): $\boldsymbol{v} = H_0 \boldsymbol{r}$.

Hubble obtained H_0=464 km/s/Mpc.

Linear relation on large scales



Kirshner, Robert P. (2004) Proc. Natl. Acad. Sci. USA 101, 8-13

Type Ia SNe allow us to extend the measurements beyond 30,000 km/sec and provide a dramatic confirmation of the Hubble law

Evolving Universe!

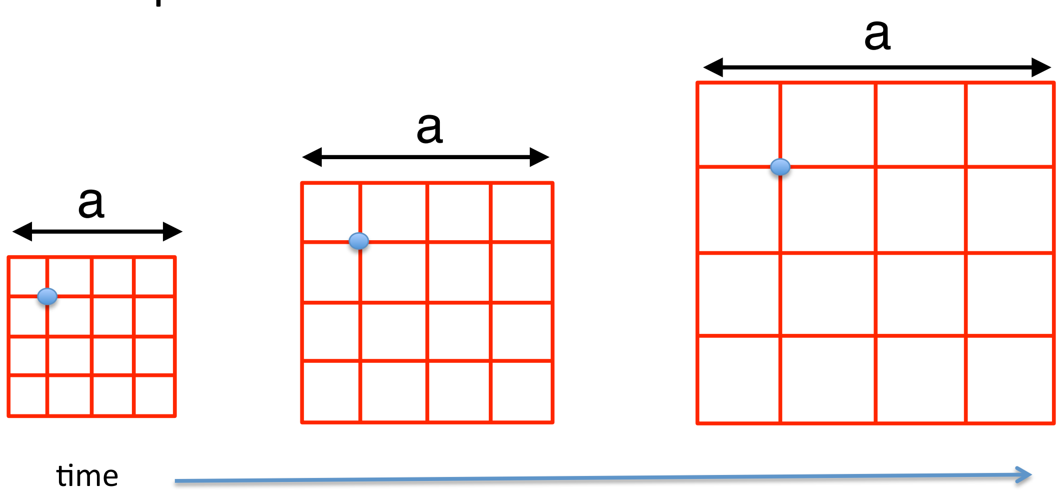
The positive value for H_0 implies the Universe is expanding, which is a natural consequence of General Relativity.

The Universe was smaller in the past, and the physical conditions must have differed; observations of distant objects show a different epoch.

The Universe is evolving.

Implication of Hubble's discovery

The Hubble law defines a special frame of reference at any point in the Universe. A **comoving** observer is at rest in this special frame of reference.



scale factor

$$\mathbf{r}(t) = a(t)\mathbf{r}(t_0)$$

$$\Rightarrow \qquad \mathbf{v}(t) = \frac{d\mathbf{r}(t)}{dt} = \frac{da(t)}{dt}\mathbf{r}(t_0)$$

$$a(t_0) = 1$$

Age of the Universe

Note that the Hubble constant has dimension 1/time:

$1/H_0$=(978 Gyr)/(H_0 in km/s/Mpc)

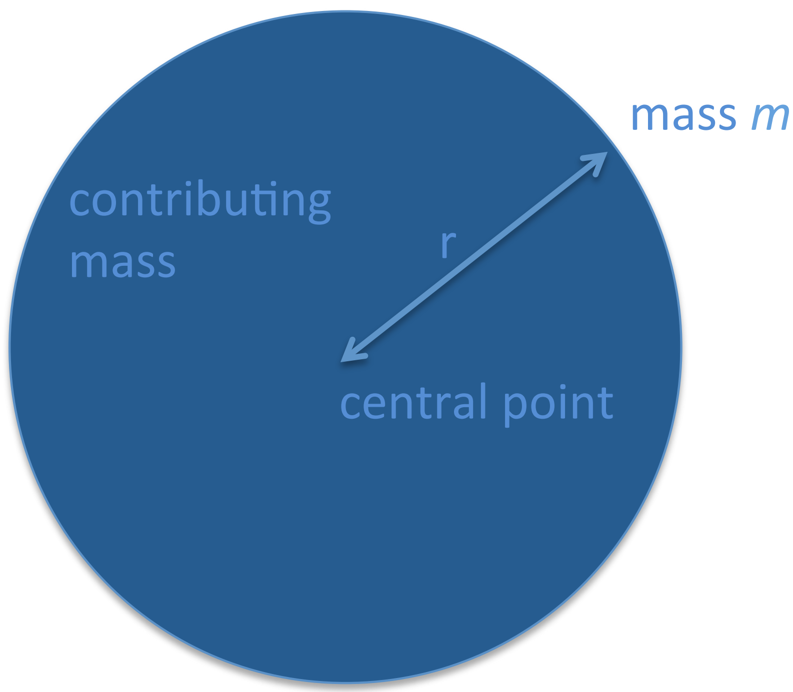
If the expansion velocity is constant then all galaxies were in the same place $t=1/H_0$ years ago. With the current estimates of $H_0=71\pm3$ km/s this implies an age of 14 Gyr.

The expansion rate is not a constant, so the result is only an indication: typically the age of the Universe is $<1/H_0$.

Similarly c/H_0 is a natural scale for the Universe.

The expanding Universe

It is possible to derive the equations that describe an expanding homogeneous Universe using Newtonian gravity. Note that a correct physical interpretation does require General Relativity.



A particle feels no force from material at radii $>r$ and the mass inside r can be treated as if all concentrated in the central point.

Friedmann equation

We used Newtonian mechanics to derive one of the Friedmann equations:

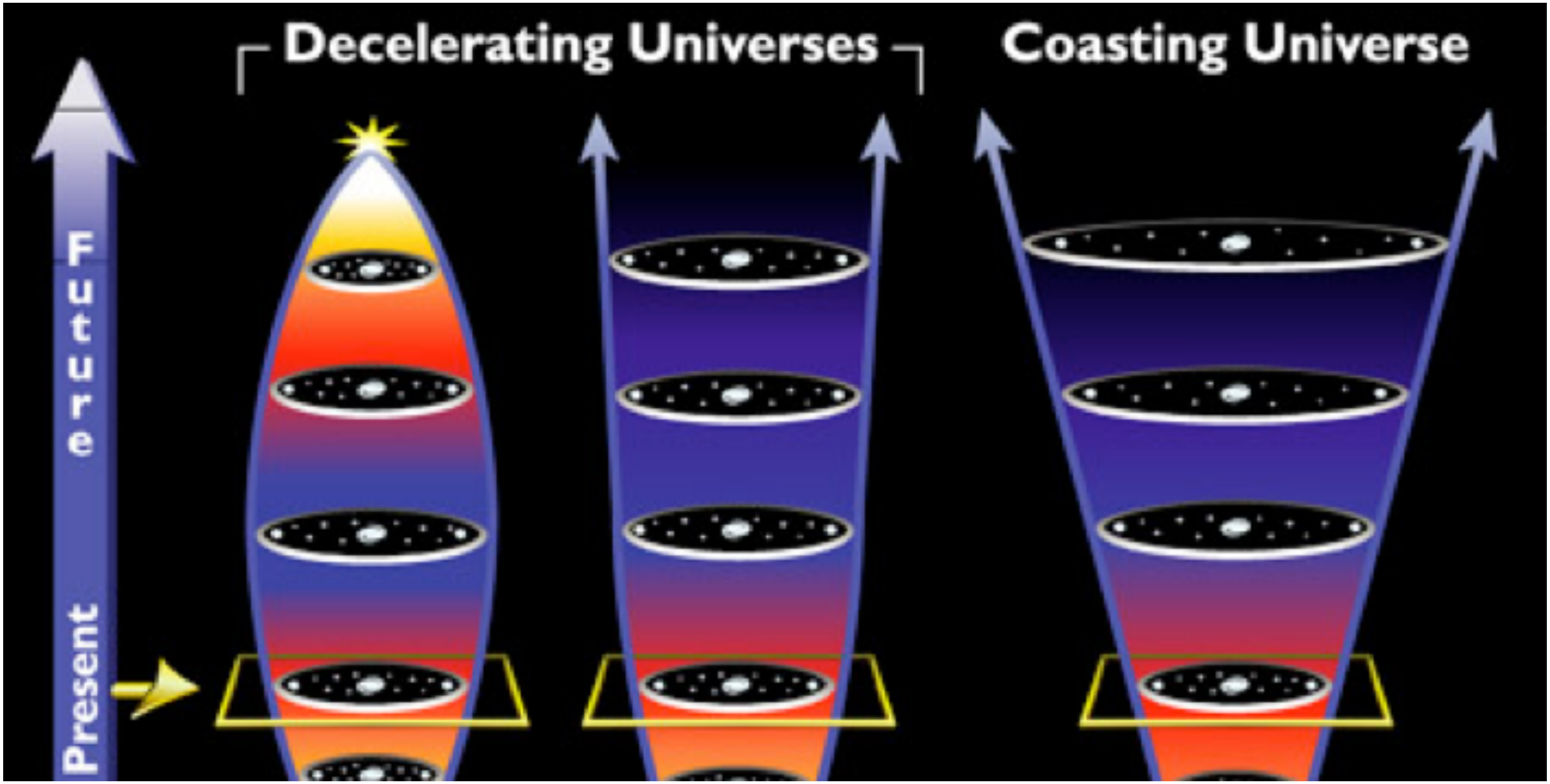
$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho - \frac{kc^2}{a^2}$$

a = scale factor for the universe

k ∝ "total energy density of universe"

Gives us an equation to calculate how the scale factor *a* of the universe evolves with time!

The fate of the Universe



What sets the value of k?

Finish material on Friedman's equation from last lecture

Friedmann equation

Here we have an equation we can use to calculate how the scale factor *a* of the universe evolves with time!

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho - \frac{kc^2}{a^2}$$

Three cases:

$k<0$: expansion never stops, as $a \to \infty$, $da/dt > 0$

$k=0$: assymptotically slow expansion, as $a \to \infty$, $da/dt \to 0$

$k>0$: expansion stops and recollapses, as $a \to \infty$, $da/dt < 0$

with $a_{max} = (8\pi G\rho_0/3kc^2)^{1/2}$

Expanding Universe

How does the scale factor a evolve with time?

Consider the simplest case k = 0 and start with Friedmann's equation:

$$((da/dt)/a)^2 = 8\pi G\rho / 3$$

Replace ρ with current density of matter in universe $\rho_{matter,0}$ and account for change in volume (divide by a^3):

$$((da/dt)/a)^2 = 8\pi G\rho_{matter,0} / 3a^3$$

Multiply by a^2 and take square root:

$$da/dt = (8\pi G\rho_{matter,0} / 3a)^{1/2}$$

$$da\ a^{1/2} = (8\pi G\rho_{matter,0} / 3)^{1/2}\ dt$$

$$\int da\ a^{1/2} = \int (8\pi G\rho_{matter,0} / 3)^{1/2}\ dt$$

$$(2/3)a^{3/2} = (8\pi G\rho_{matter,0} / 3)^{1/2}\ t$$

$$a = t^{2/3}\ (6\pi G\rho_{matter,0})^{1/3}$$

If we take $t_0 = (1/6\pi G\rho_{matter,0})^{1/2}$, then

$$a = (t/t_0)^{2/3}$$

Using this convention, a = 1 when $t = t_0$, i.e., the present

Expanding Universe

What is the age of the universe?

From the previous page: $t_0 = (1/6\pi G\rho_{matter,0})^{1/2}$

We commented previously that the age of the universe was roughly equal to $1/H_0$ — i.e. the reciprocal of the Hubble parameter

Let's rewrite the above expression using the Hubble parameter

Hubble constant defines relationship between distance and velocity:

$$v = Hd \longleftrightarrow dr/dt = Hr$$

$$x\,da/dt = Hxa$$

$$H = (da/dt)/a$$

Rewrite Friedmann's equation as $H^2 = 8\pi G\rho / 3$

Solve for ρ: $\rho = 3H^2/8\pi G / 3$

If we substitute this in the above equation:

$$t_0 = 2/(3H_0)$$

So the age of the universe is of the same order as $1/H_0$ — but smaller because of deceleration in the expansion from gravity.

Expanding Universe

Note that the $k = 0$ falls exactly between the $k < 0$ and $k > 0$ cases

The density of the universe in this case is the critical density.

$$\rho = 3H^2/8\pi G / 3 = \rho_{critical}$$

Because of the relevance of the critical density in terms of interpreting the expansion of the universe, we often use the quantity Ω to define the density relative to critical:

$$\Omega(a) = \rho/\rho_{crit}$$

We will later rewrite Friedmann's equation in terms of Ω

Deceleration

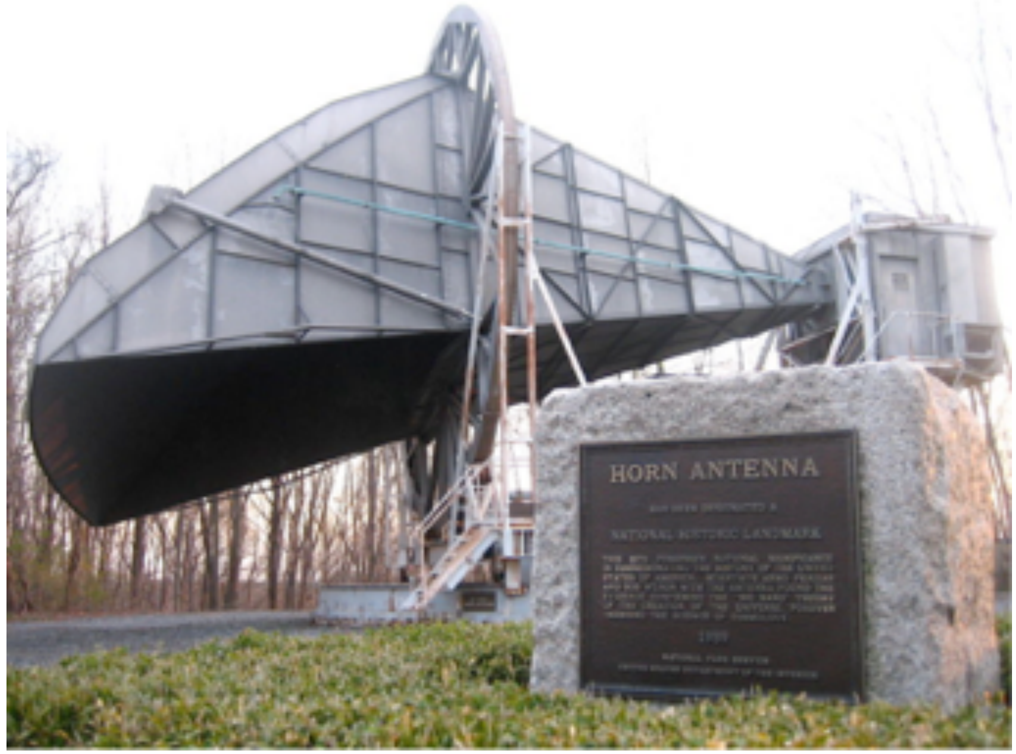
The expansion rate is expected to slow down due to the gravity of matter in the Universe. Therefore by measuring the rate of change we can “weigh” the Universe.

Therefore the Hubble constant is not constant, but varies with time: it is better to talk about the **Hubble parameter**.

To measure the deceleration, we need to extend the measurements to much larger distances, which became possible in the 1990s using type Ia supernovae.

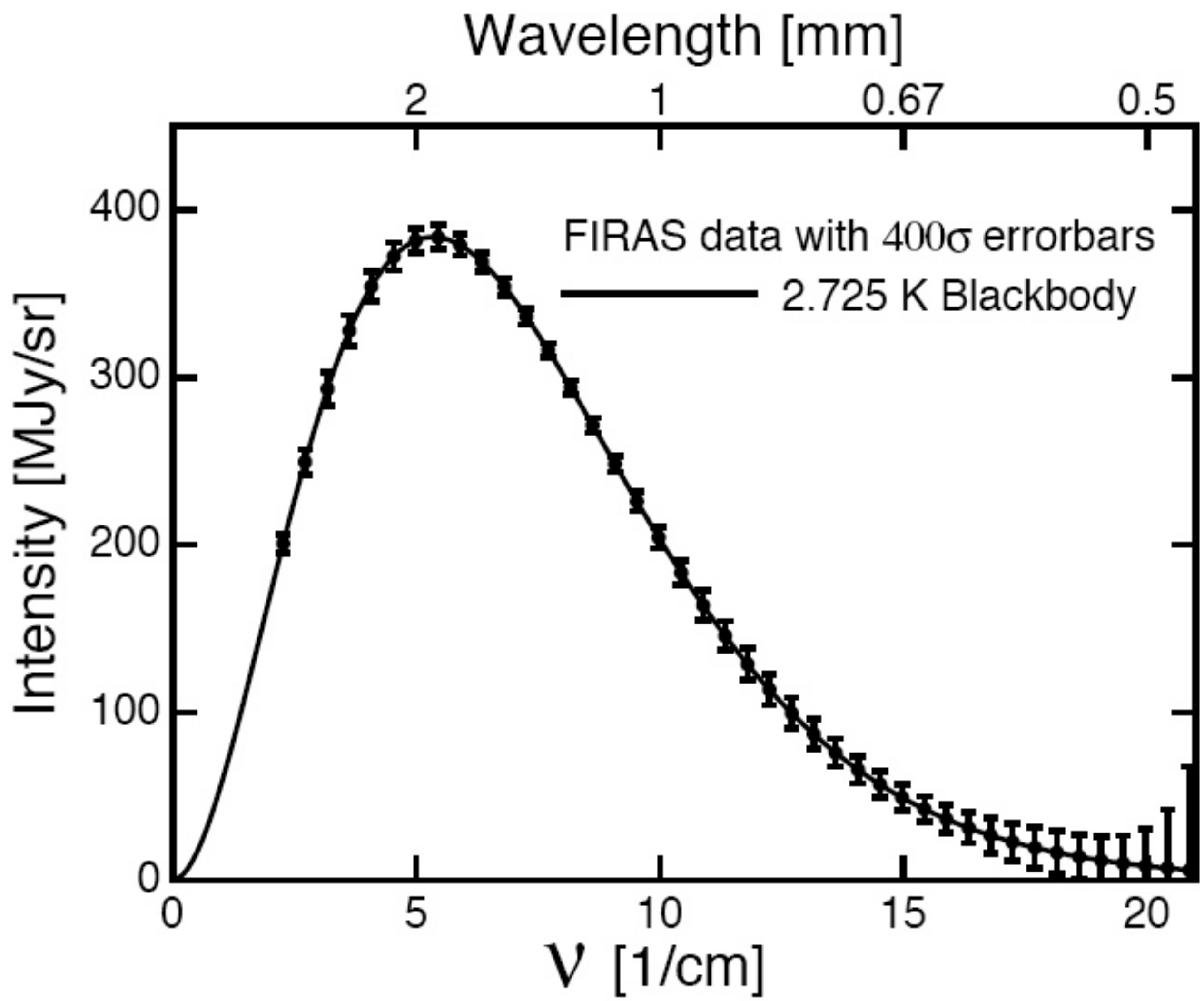
Evidence for Homogeneity and Isotropy

Cosmic Microwave Background



In 1964 Penzias & Wilson detected radio noise that came from all directions. This discovery made cosmology a genuine science topic! It has developed into the prime tool for precision cosmology and the measurements keep improving .

Perfect Black Body?



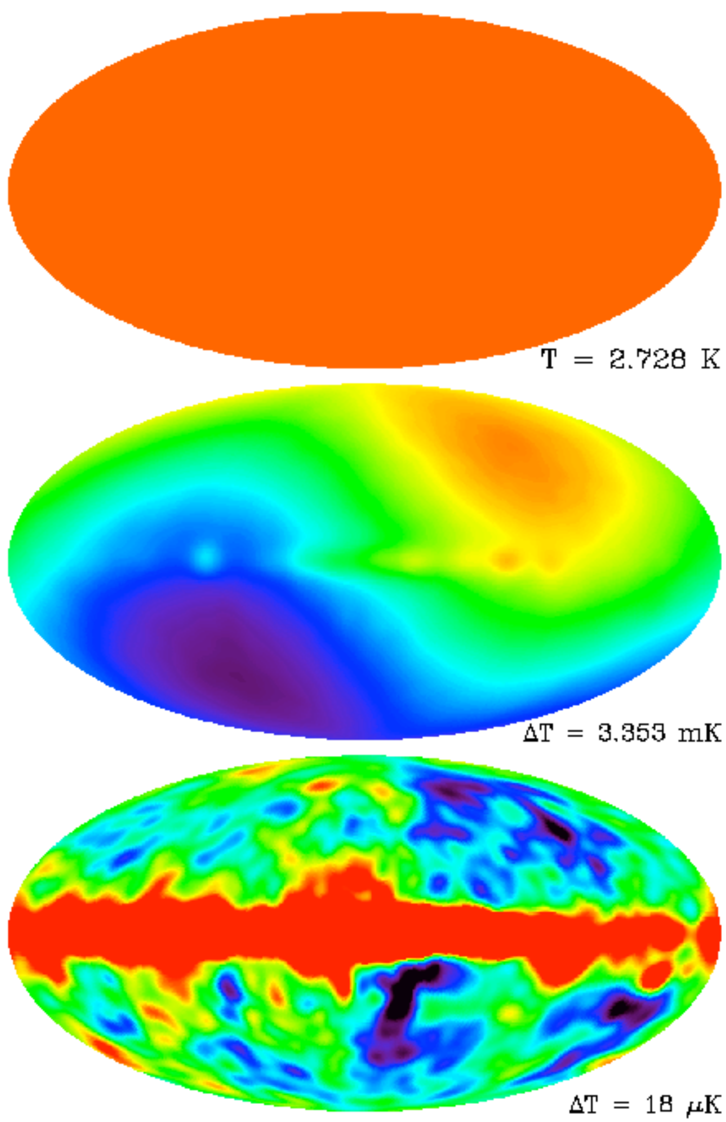
The CMBR emission peaks in mm-wave part of spectrum and is (still) the most precisely measured perfect black body with T = 2.72548±0.00057 K.

Cosmic Microwave Background

In the past the Universe was much smaller, and much denser and hotter. At some point it was so hot that hydrogen was ionized and the mean free path of photons was short: plasma in thermal equilibrium.

This is a prediction of the hot Big Bang model, which could rule out the steady-state model which was its main competitor.

The CMB seen by COBE

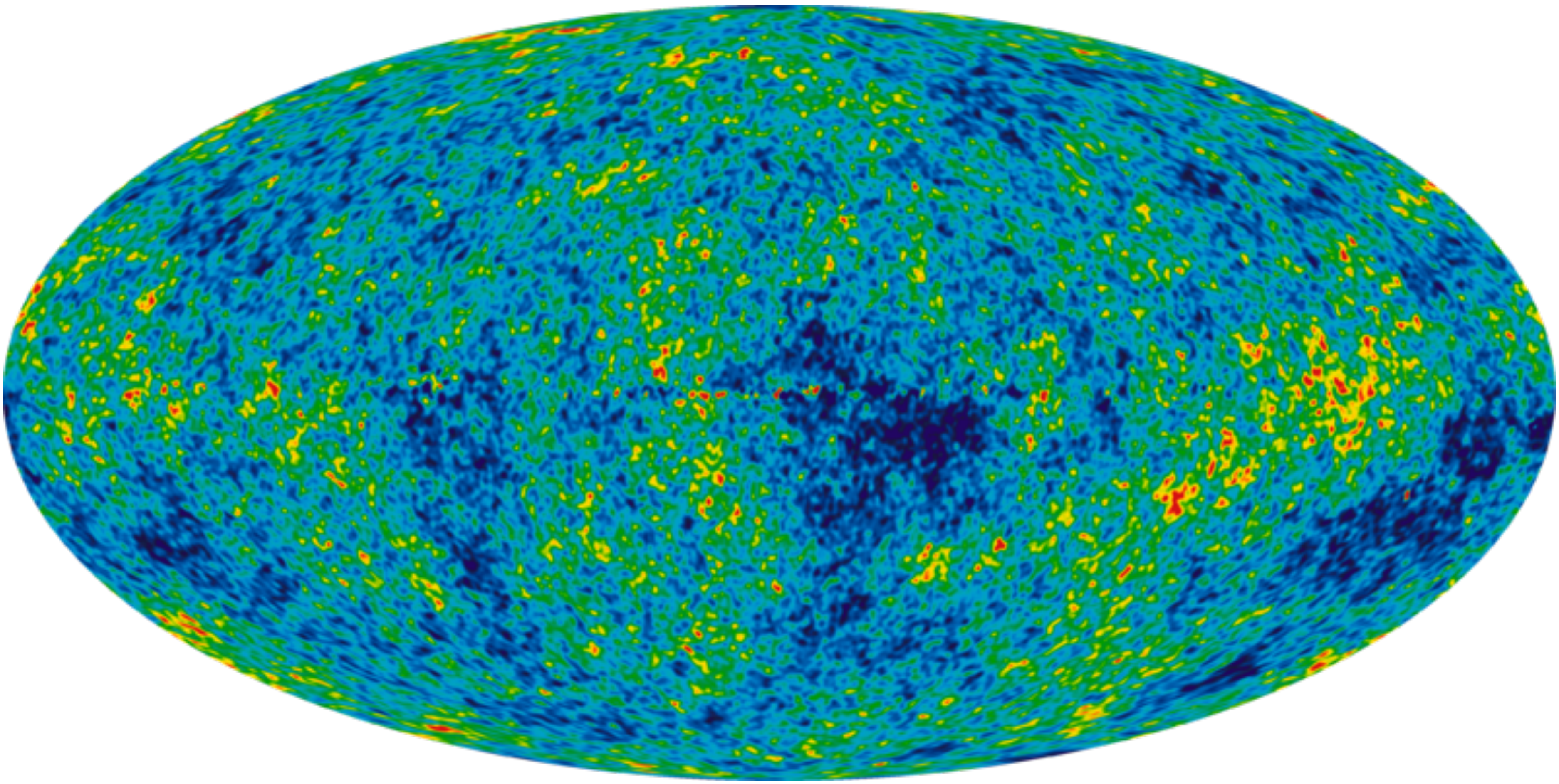


The CMB is extremely uniform

The main feature is the dipole, due to our peculiar motion: the solar system is moving at 365 km/s w.r.t. the CMB

It also shows very small temperature fluctuations on the order of 1 part in 100,000 across the sky. Note the contamination by our galaxy.

The CMB seen by WMAP



Credit: NASA / WMAP Science Team

Distribution of galaxies

We can measure the number of galaxies in different directions and at different brightness in the sky. When a distribution is the same in all directions, it is **isotropic**.

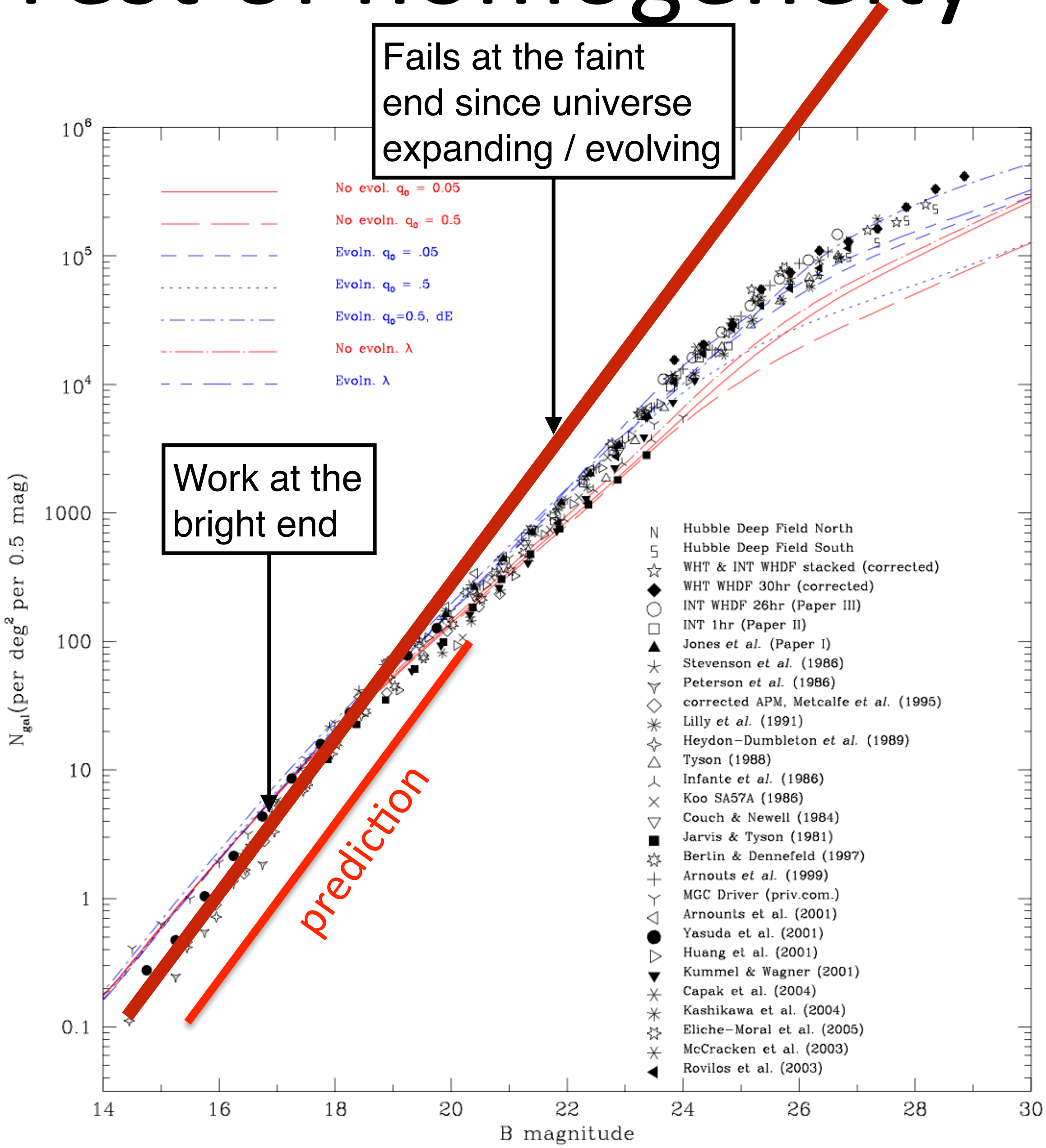
Consider a homogeneous distribution of identical sources:

- apparent flux at distance *r*: $S \propto r^{-2}$
- number of sources within distance *r*: $N \propto r^{3}$

⇒ the number of sources brighter than flux *S*: $N(>S) \propto S^{-3/2}$
(a source that is 100x fainter is 10x further away, an the volume is 1000x larger)

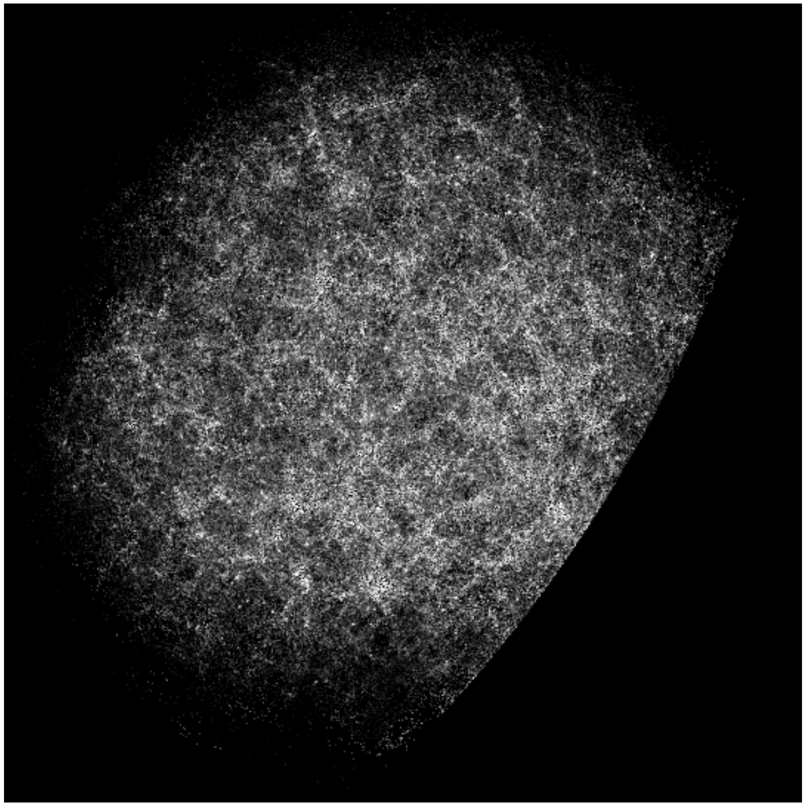
This can be generalized to a distribution of source luminosities

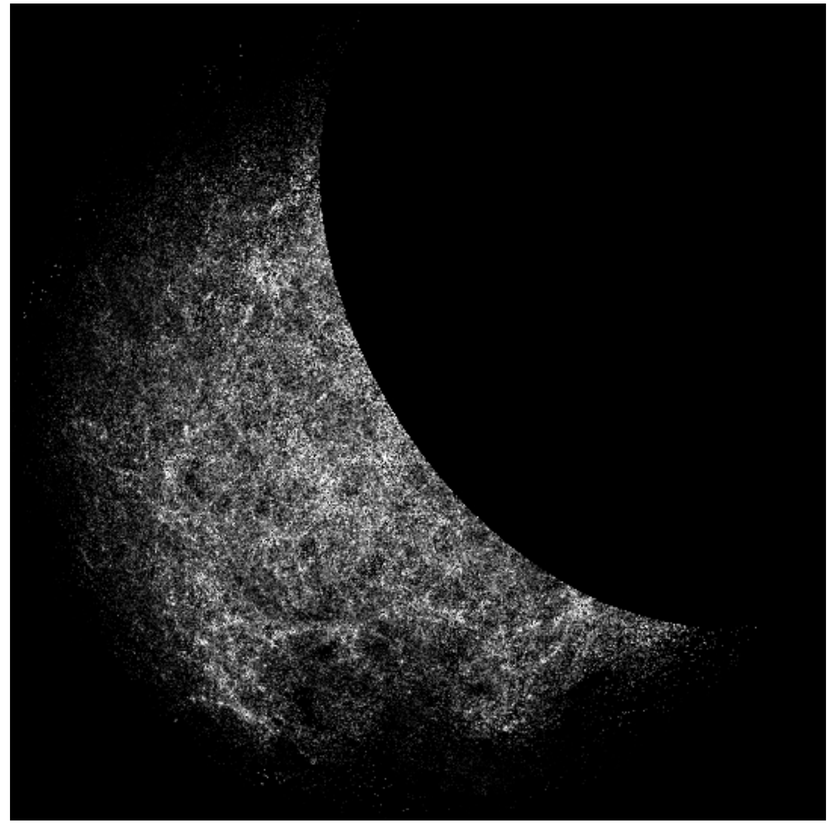
Test of homogeneity



Courtesy Nigel Metcalfe

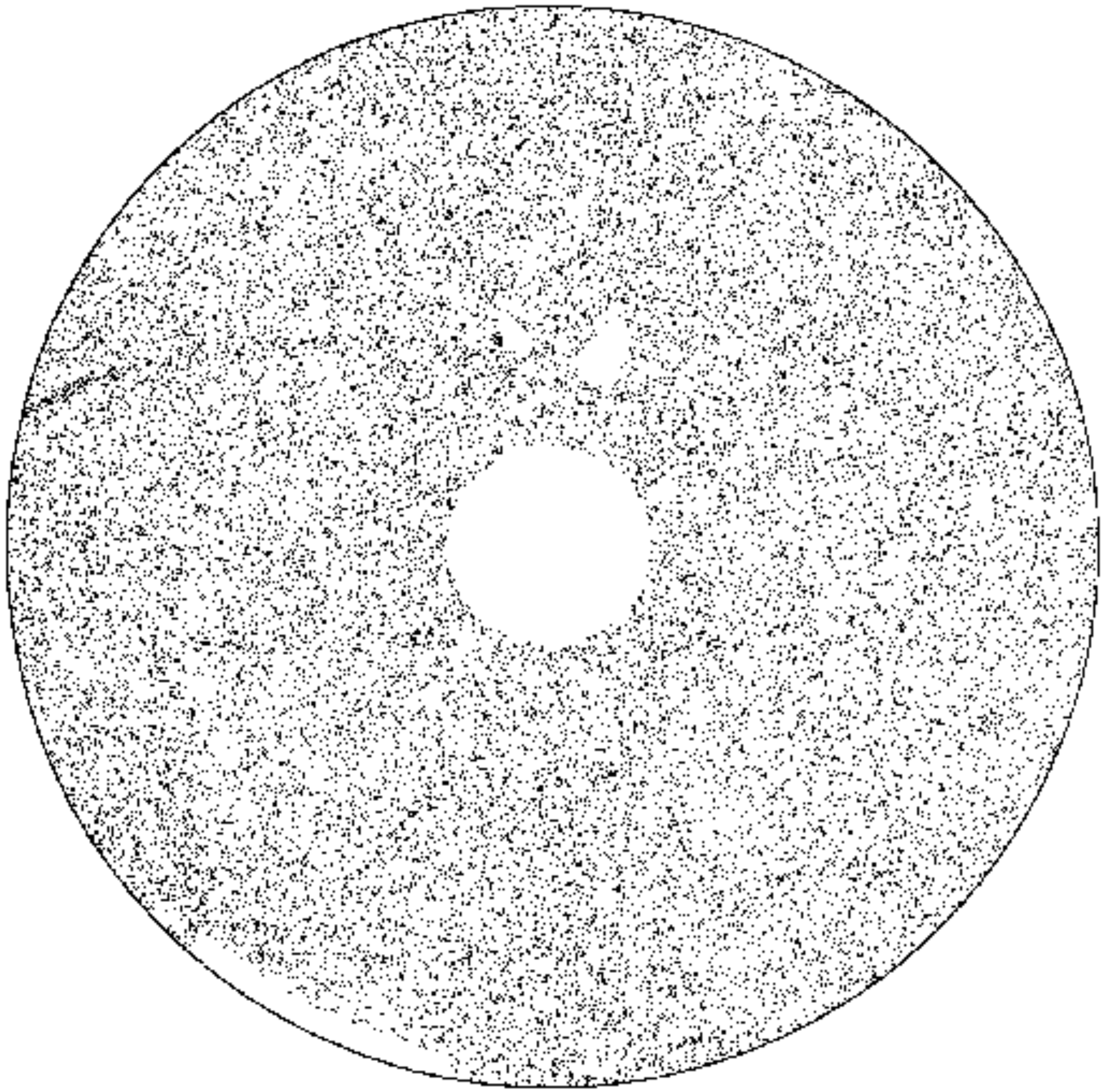
Distribution of nearby galaxies





The angular distribution of galaxies brighter than B~19 from the Lick galaxy catalog shows a lot of clustering.

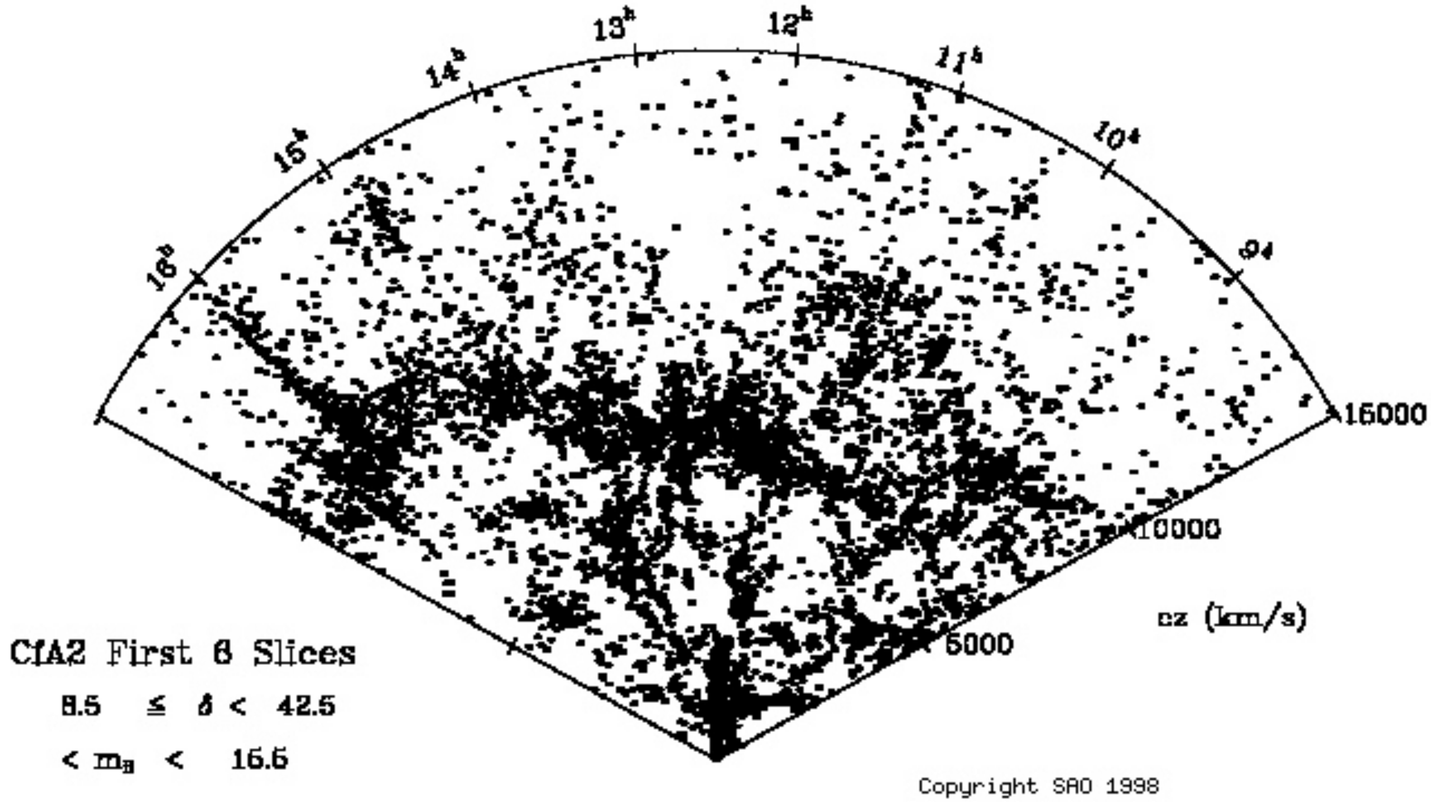
Distribution of radio sources



Gregory & Condon (1991): angular distribution of the 31,000 brightest 6cm radio sources

Redshift surveys

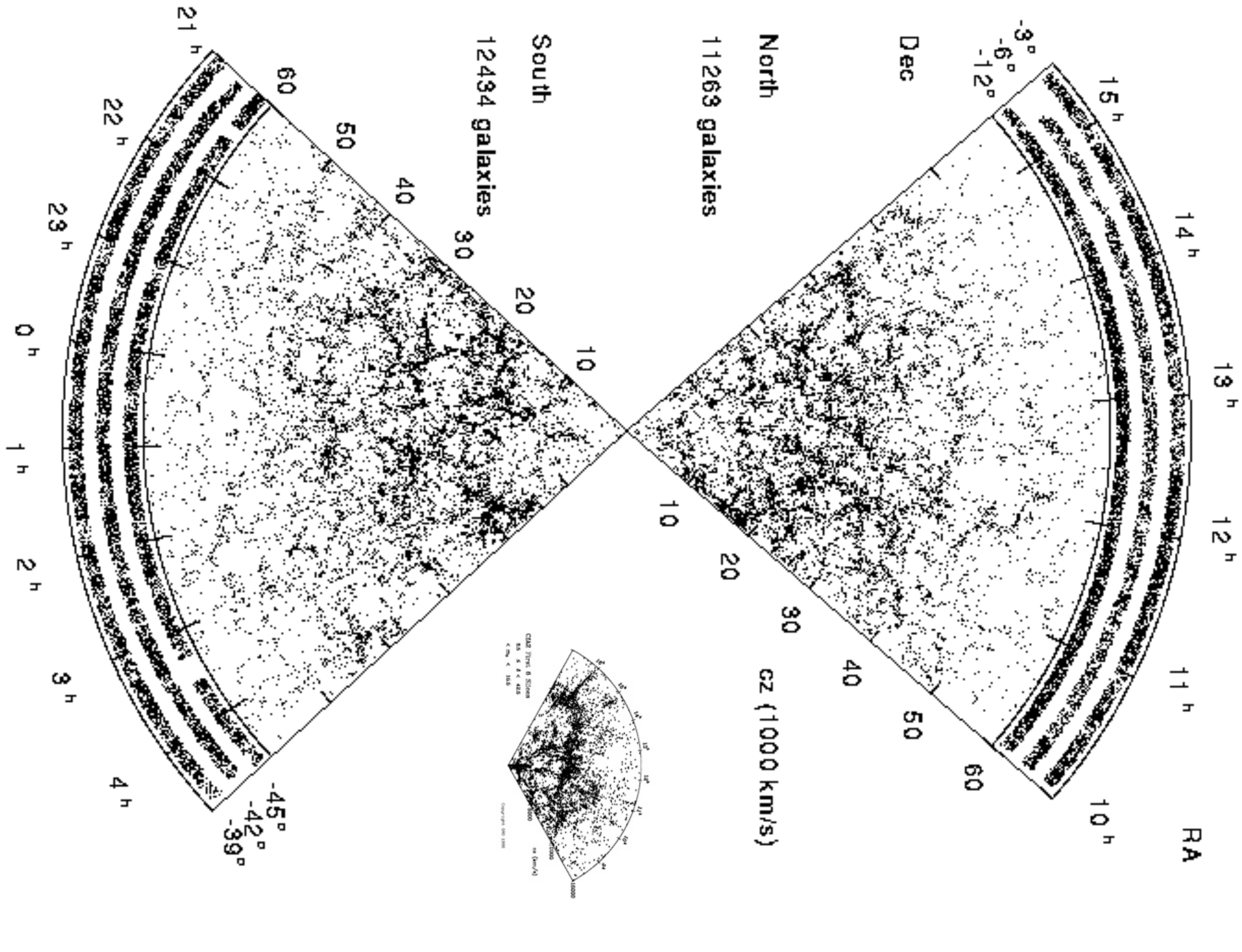
If the distances are large enough we can use the Hubble law to estimate distances: $r=v/H_0$.



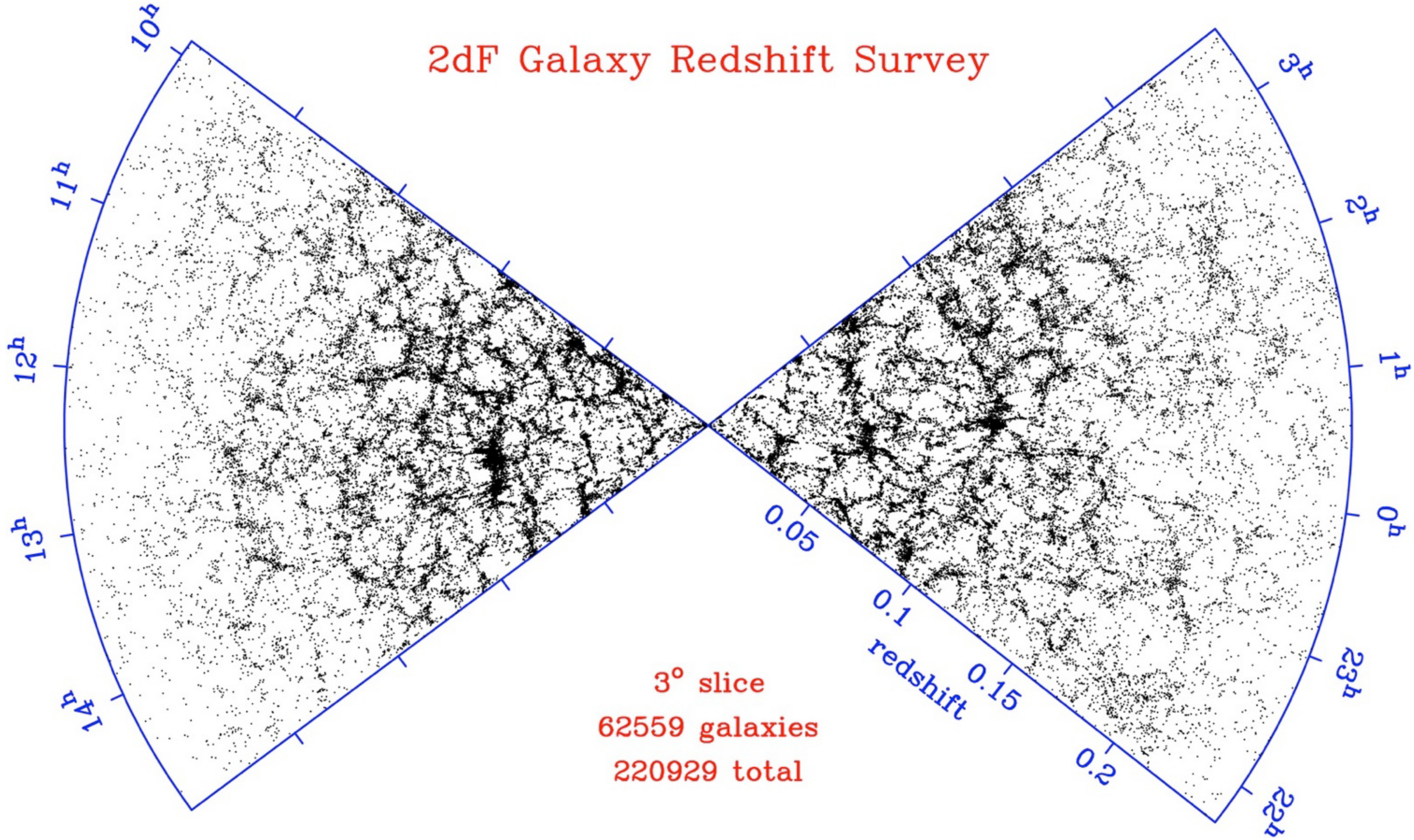
The CfA redshift survey targeted 18,000 redshifts and took almost 10 years to complete.

Bigger redshift surveys

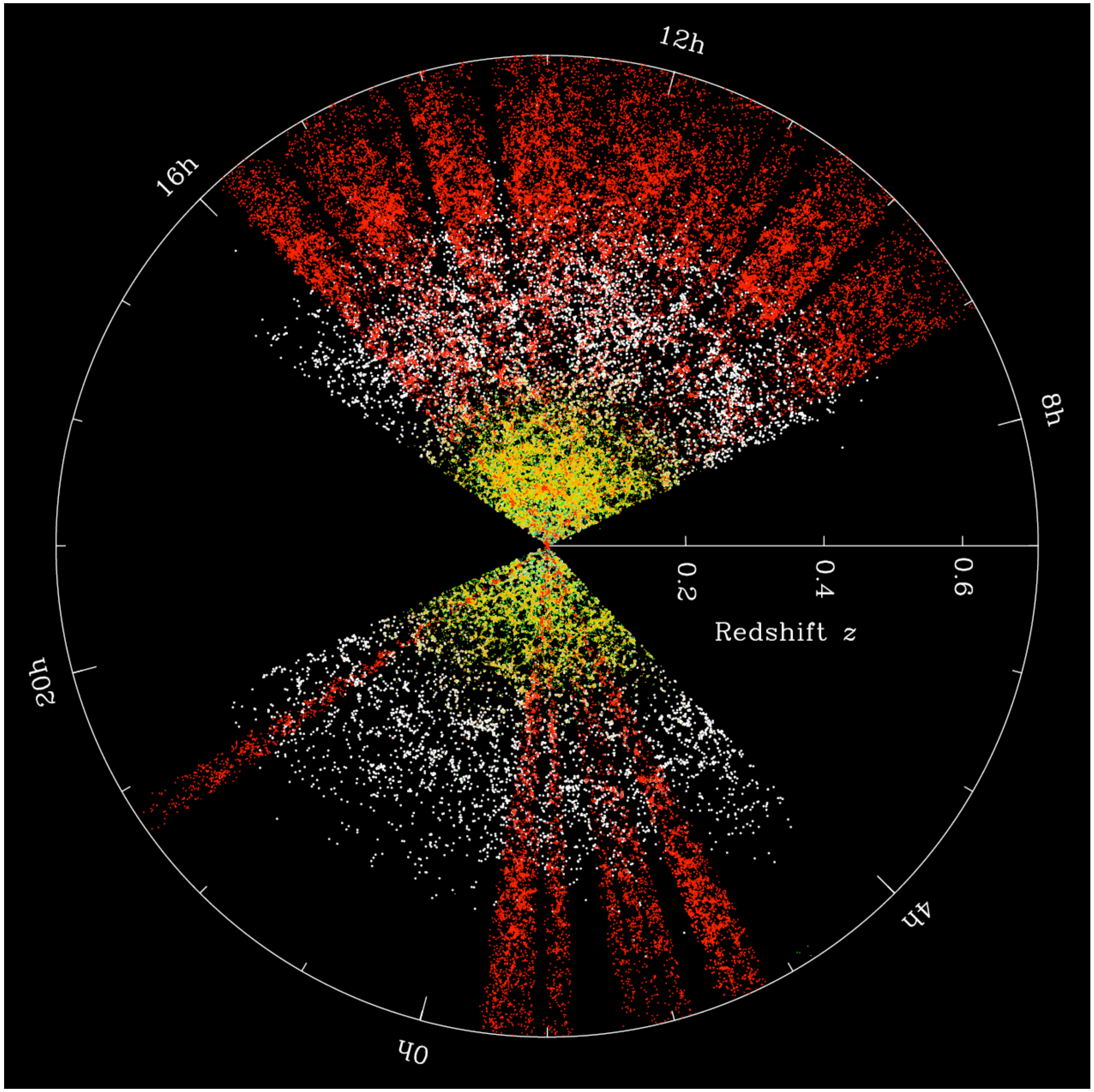
The Las Campanas redshift survey (26418 redshifts out to z~0.2) was the first to indicate homogeneity on large scales.



Even larger redshift surveys

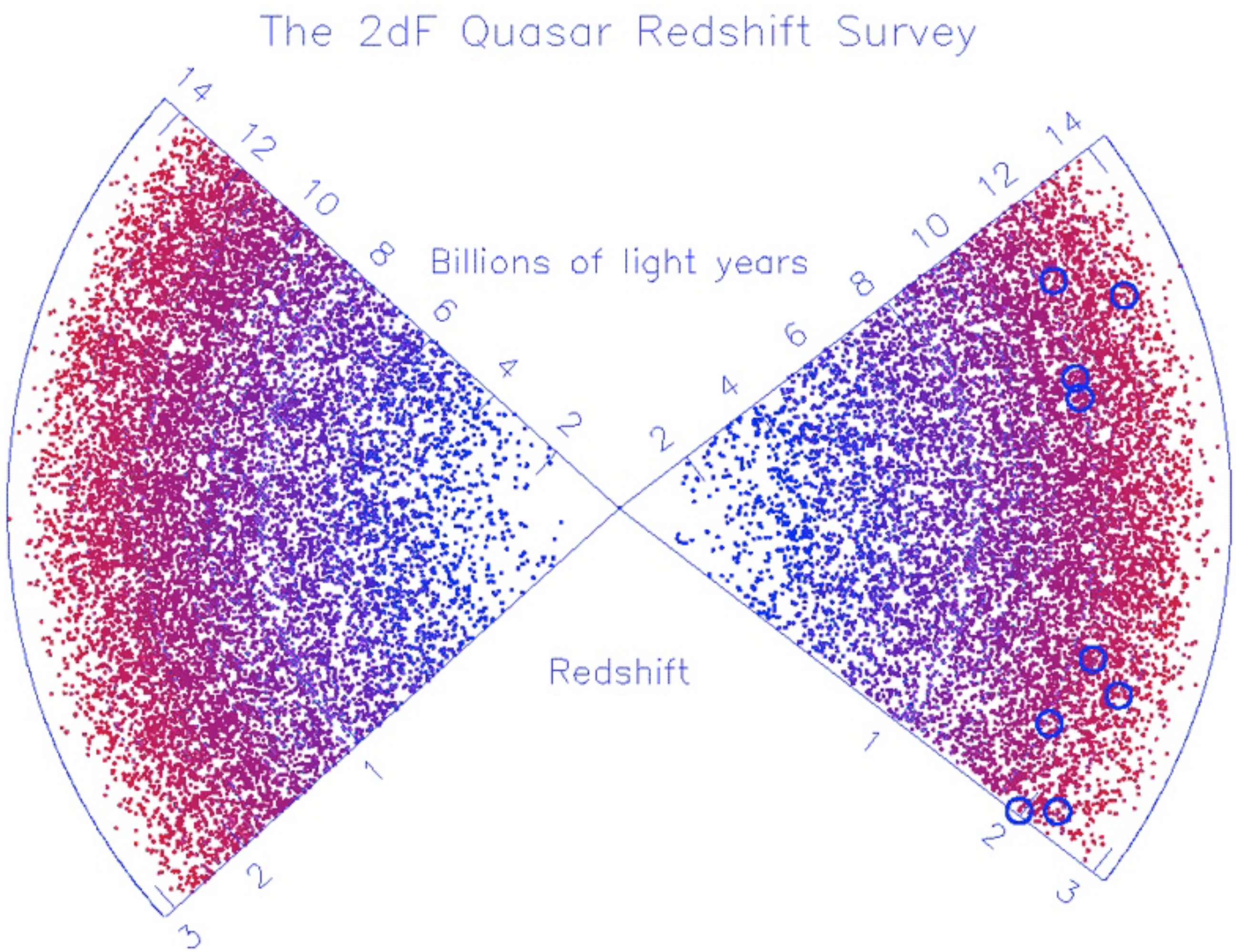


Even larger redshift surveys



SDSS-III (BOSS)

Look at distant objects



More distant objects are more homogeneous. These results also support **isotropy**.

Homogeneity

On small scales the distribution of galaxies is “foam”-like, and we will explore the basic principle behind this later in this course.

The 2dF galaxy redshift survey and the even larger Sloan Digital Sky Survey clearly demonstrated **statistical** homogeneity.

Fluid and Acceleration Equations

How does the density evolve?

Apply first law of thermodynamics to expanding universe:

$$dE + PdV = TdS$$

Expanding volume of unit comoving radius

Volume has physical radius of a, so

$$E = mc^2 = (4\pi/3)a^3\rho c^2$$

$$dE/dt = 4\pi a^2\rho c^2\, da/dt + (4\pi/3)a^3c^2\, d\rho/dt$$

$$dV/dt = 4\pi a^2\, da/dt$$

According to the cosmological principle, the universe is on the whole homogeneous and so there is no net flow of heat or entropy => dS = 0

$$dE/dt + PdV/dt = 0$$

$$3\rho(da/dt)/a + d\rho/dt + 3P(da/dt)/a/c^2 = 0$$

$$d\rho/dt + 3((da/dt)/a)(\rho + P/c^2) = 0$$

How does the density evolve?

$$d\rho/dt + 3((da/dt)/a)(\rho + P/c^2) = 0$$

Note there are no pressure forces, as these require gradients.

We need an equation of state $P = P(\rho)$ Equation depends on material

Baryonic material has an equation of state $P = w\rho$, where $w \sim 0$

$P = kT (\rho/\mu)$ where μ is mean molecular mass

Energy density $\varepsilon = \rho c^2$ $P = (kT/\mu c^2)\varepsilon$

For a non-relativistic gas, $3kT = \mu\langle v^2\rangle$

So, $w = P/\varepsilon = (1/3)(\langle v^2\rangle/c^2) \ll 1$

Photons and highly relativistic particles have $w = 1/3$

w cannot take on arbitrary values: variations in P will travel at the sound speed $c_s^2 = c^2(dP/d\rho)_s$

$c_s = c\, w^{1/2}$

How does the density evolve?

We can combine the Friedmann's equation with the fluid equation to derive a formula describing the acceleration of the scale factor:

$$(d/dt)((da/dt)/a)^2 = (d/dt)(8\pi G\rho/3 - kc^2/a^2)$$

$$2((da/dt)/a)((d^2a/dt^2)/a - (da/dt)^2/a^2) = 8\pi G(d\rho/dt)/3 + 2(da/dt)kc^2/a^3$$

Use the fluid equation $d\rho/dt + 3((da/dt)/a)(\rho + P/c^2) = 0$

$$(d^2a/dt^2)/a - (da/dt)^2/a^2 = -4\pi G(\rho + 3P/c^2) + kc^2/a^2$$

$$(d^2a/dt^2)/a = 8\pi G\rho/3 - 4\pi G(\rho + P/c^2) = 4\pi G(\rho/3 + P/c^2)$$

$$(d^2a/dt^2)/a = (-4\pi G/3)(\rho + 3P/c^2)$$

$$(d^2a/dt^2)/a = (-4\pi G/3c^2)(\varepsilon + 3P)$$

ASIDE: Particles in the Universe

For a particle with rest mass m: $E^2 = m^2c^4 + p^2c^2$

If the mass-energy dominates, the particle will be moving at much less than the speed of light, and is said to be non-relativistic.

Relativistic: photons and neutrinos
Non-relativistic: baryons and dark matter (?)

How does the density evolve?

Universe is filled with different components:

$$\varepsilon = \Sigma_w \ \varepsilon_w$$

Fluid equation holds for each (assuming no interaction):

$$d\varepsilon_w/dt + 3((da/dt)/a)(\varepsilon_w + P_w) = 0$$

Implies

$$\varepsilon_w = \varepsilon_{w,0} \, a^{-3(1+w)}$$

Radiation Component Scales as $1/a^3 \cdot 1/a$ $\Rightarrow w = 1/3$

$$\rho_{rad} = \rho_{rad,0} / a^4$$

How does the density evolve?

For $\varepsilon_w = \varepsilon_{w,0}\, a^{-3(1+w)}$

Matter Component Scales as $1/a^3$ $\Rightarrow w = 0$

$$\rho_{matter} = \rho_{matter,0} / a^3$$

Dark Energy Component is Constant $\Rightarrow w = -1$

$$\rho_\Lambda = \rho_{\Lambda,0}$$

Will motivate Dark Energy properties later

How does the density evolve?

Let's recast the Friedmann equation in terms of the critical density $\rho_{critical} = 3H_0^2/8\pi G$:

Start with Friedman's Equation: $((da/dt)/a)^2 = (8\pi G\rho/3 - kc^2/a^2)$

Manipulate it:

$$((da/dt)/a)^2 = (8\pi G(\rho_{matter}+\rho_{rad}+\rho_{\Lambda})/3 - kc^2/a^2)$$

$$((da/dt)/a)^2 = H_0^2(8\pi G(\rho_{matter}+\rho_{rad}+\rho_{\Lambda})/3H_0^2 - kc^2/a^2H_0^2)$$

$$((da/dt)/a)^2 = H_0^2(8\pi G\rho_{matter}/3H_0^2 + 8\pi G\rho_{rad}/3H_0^2 + 8\pi G\rho_{\Lambda}/3H_0^2 - kc^2/a^2H_0^2)$$

$$((da/dt)/a)^2 = H_0^2(8\pi G\rho_{matter,0}/3H_0^2a^3 + 8\pi G\rho_{rad,0}/3H_0^2a^4 + 8\pi G\rho_{\Lambda,0}/3H_0^2 - kc^2/a^2H_0^2)$$

How does the density evolve?

$$((da/dt)/a)^2 = H_0^2(8\pi G\rho_{matter,0}/3H_0^2a^3 + 8\pi G\rho_{rad,0}/3H_0^2a^4 + 8\pi G\rho_{\Lambda,0}/3H_0^2 - kc^2/a^2H_0^2)$$

Define $\Omega = \rho\ /\ \rho_{crit}$

$$((da/dt)/a)^2 = H_0^2(\rho_{matter,0}/\rho_{critical}\ a^3 + \rho_{rad,0}/\rho_{critical}\ a^4 + \rho_{\Lambda,0/}/\rho_{critical} - kc^2/a^2H_0^2)$$

Define $\Omega_k = -\ kc^2/H_0^2$

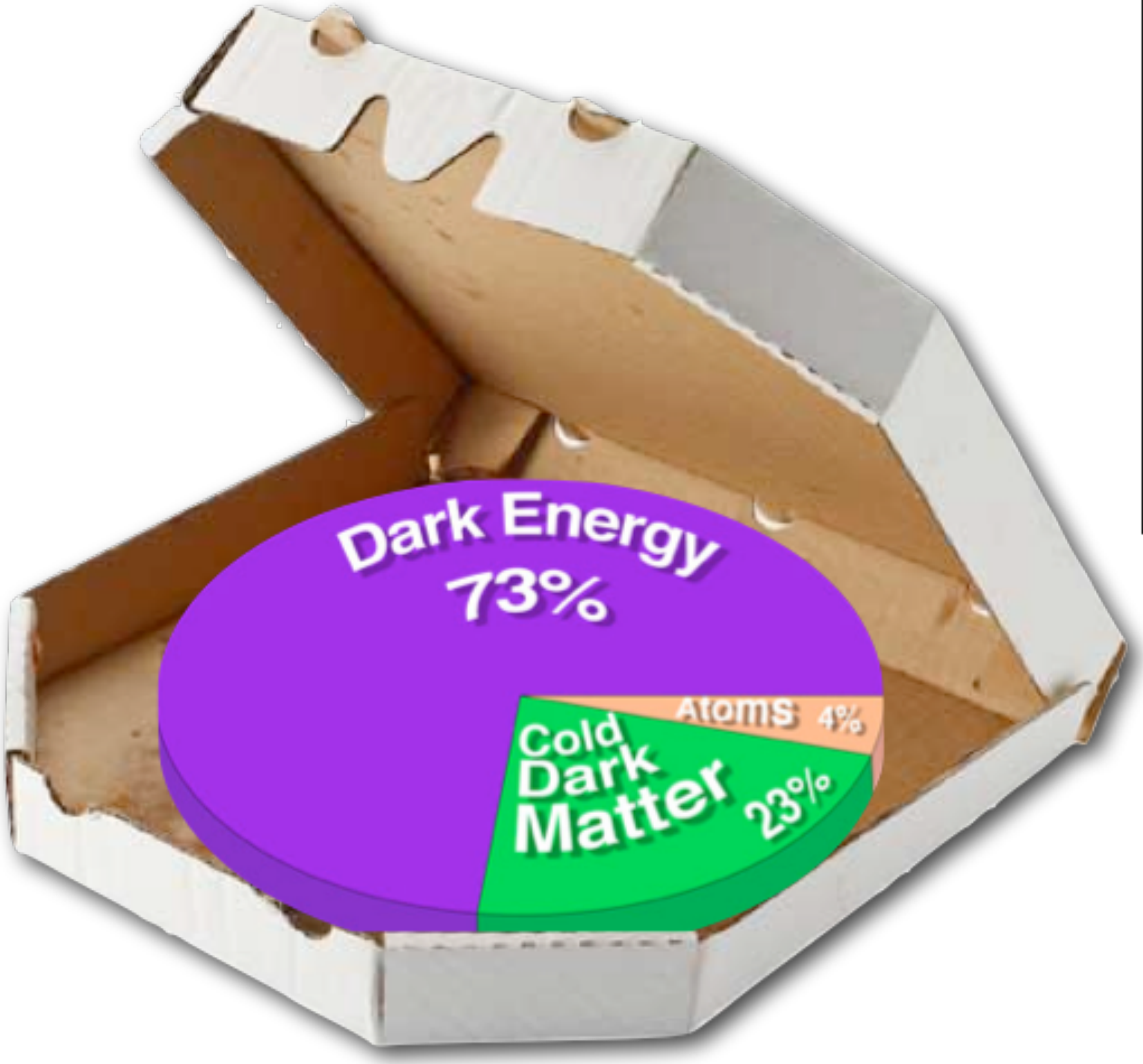
$$((da/dt)/a)^2 = H_0^2(\Omega_{matter,0}\ /\ a^3 + \Omega_{rad,0}\ /\ a^4 + \Omega_{\Lambda,0} + \Omega_{k,0}/a^2)$$

Take $H(a) = (da/dt)/a$

Another form of Friedman's equation

$$H^2(a) = H_0^2(\Omega_{matter,0}\ /\ a^3 + \Omega_{rad,0}\ /\ a^4 + \Omega_{\Lambda,0} + \Omega_{k,0}/a^2)$$

The standard model of cosmology



Ingredients
(per Universe)

DE: 73%
CDM: 23%
H: 3%
He: 1%
neutrinos: 0.3%