Sterrewacht Leiden

Origins & Evolution of the Universe

an introduction to cosmology — Fall 2018

Lecture 12: Finishing Thoughts / Review

Rychard Bouwens

Layout of the Course

Sep 24: Introduction and Friedmann Equations
Oct 1: Fluid and Acceleration Equations
Oct 8: Introductory GR, Space Time Metric, Proper Distance
Oct 15: Redshift, Horizons, Observable Distances
Oct 17: Problem Class #1
Oct 22: Observable Distances, Parameter Constraints
Oct 29: Thermal History, Early Universe
Nov 5: Early Universe, Inflation
Nov 12: Inflation, Lepton Era, Big Bang Nucleosynthesis
Nov 14: Problem Class #2
Nov 19: Recombination, Cosmic Microwave Background Radiation
Nov 26: CMB Radiation (II), Introduction to Structure Formation
Dec 3: Introduction to Structure Formation (II)
Dec 5: Problem Class #3
Dec 10: Finishing Thoughts, Review
Dec 21: Final Exam

Review Last Week

Modeling the Growth of Structure using Waves in Fluid

We will look for solution in the form of plane waves $\delta u_i = \delta_i \, e^{i\mathbf{k} \cdot \mathbf{r}}$ where $\delta u_i = \delta\rho, \delta v, \delta\phi, \delta s$

Given that the unperturbed solution do not depend on position, we can search for solutions:

$\delta_i(t) = \delta_{0,i} \, e^{i\omega t}$ amplitude D,V, Φ, Σ

The solutions are of two types, depending on whether $\lambda = 2\pi/k$ larger or smaller than

$$\lambda = c_s \, (\pi/G\rho_0)^{1/2}$$

In the case that $\lambda < \lambda_J$, the value of ω is real and $\omega = \pm c_s \, k[1-(\lambda/\lambda_J)^2]^{1/2}$

These represent two sound waves in directions $\pm\mathbf{k}$ with a dispersion ω

If $\lambda > \lambda_J$, the frequency is imaginary: $\omega = \pm \, i \, (4\pi G\rho_0)^{1/2} \, [1-(\lambda_J/\lambda)^2]^{1/2}$

and the solution for the density is $d\rho/\rho_0 = \delta_0 \, e^{i\mathbf{kr}} \, e^{\pm\omega t}$

The characteristic time scale for the evolution of the amplitude is

$$\tau = \omega^{-1} = 1/(4\pi G\rho_0)^{1/2} \, [1-(\lambda_J/\lambda)^2]^{-1/2}$$

for $\lambda >> \lambda_J$, this corresponds to the dynamical or free-fall time.

Growth of Structure in Expanding Universe

Let us now look at the homogeneously expanding solution with expansion faction a(t)

$$\rho_{bg} = \rho_0\,(a/a_0)^{-3},\ \ \mathbf{v}_{bg} = ((da/dt)/a)\mathbf{r},\ \Phi_{bg} = (2/3)\pi G\rho_{bg}r^2,\ p_{bg} = p(\rho_{bg})$$

This results in the following equation:

$$\Rightarrow d^2\delta/dt^2 + 2((da/dt)/a)\ d\delta/dt + ik(c_s^2k^2 - 4\pi G\rho)\delta = 0$$

To solve this equation, we need a prescription of a, ρ, and c_s

* Flat matter-dominated Einstein-de Sitter model

$$\rho = 1/(6\pi Gt^2) \qquad a = a_0\,(t/t_0)^{2/3} \qquad (da/dt)/a = H = (2/3t)$$

$$\Rightarrow d^2\delta/dt^2 + (4/3)(d\delta/dt)/t - (2/3t^2)[1 - c_s^2k^2/4\pi G\rho]\delta = 0$$

If we assume that matter compromises monoatomic particles of mass m, then the sound speed is

$$c_s = (5k_BT_m/3m)^{1/2} = (5k_BT_{0,m}/3m)^{1/2}\,(a_0/a)$$

In this case that c_sk is very small (long wavelengths, low sound speed)

$$\Rightarrow d^2\delta/dt^2 + (4/3)(d\delta/dt)/t - (2/3t^2)\delta = 0$$

Try a solution $\delta \propto t^n$

$$\Rightarrow [n(n-1) + (4/3)n - 2/3]t^{n-2} = 0$$

$$\Rightarrow n(n-1) + (4/3)n - 2/3 = 0 \qquad \Rightarrow n=-1 \text{ or } n=2/3$$

growing mode $\delta_+ \propto t^{2/3} \propto a$; decaying mode $\delta_- \propto t^{-1}$

The densities grow $\delta \propto t^{2/3} \propto a(t) \propto 1/(1+z)$ as long as $\delta << 1$

Growth of Structure in Expanding Universe: Different cases

For large k (short wavelength) and under the assumption that c_s varies slowly

$$d^2\delta/dt^2 + (4/3)(d\delta/dt)/t - (2/3\rho)(1 - c_s^2k^2/4\pi G)\delta = 0$$

If we try again $\delta \propto t^n$ we find solutions $n^2 + (n/3) - (2/3)[1 - c_s^2k^2/4\pi G\rho] = 0$

$$n = -(1/6) \pm (1/6)(25 - 6c_s^2k^2/\pi G\rho)^{1/2} = 0$$

hence instability when $k <\sim (G\rho)^{1/2}/c_s^2$ and oscillations for larger k

If we consider a low Ω_m where curvature dominates, then $a \propto t$ and

$d^2\delta/dt^2 + 2d\delta/dt/t = 0$, which has solutions $\delta \propto t^{-1}$ and $\delta \propto t^0$

i.e., no growth in a low density universe

If we consider a lambda-dominated universe, then

and $d^2\delta/dt^2 + 2(da/dt/a)\, d\delta/dt = 0$, with solutions $\delta_- \propto e^{-2Ht}$ and $\delta_+ \propto t^0$

i.e., no growth in a lambda-dominated universe

In the case of a radiation dominated universe, the derivation needs to include the pressure in the energy density $\rho \rightarrow \rho + P/c^2$ and one can show that

$$d^2\delta/dt^2 + 2((da/dt)/a)d\delta/dt + [c_sk^2 - 32/3\pi\, G\rho]\delta = 0$$

For a radiation dominated universe where $a \propto t^{1/2}$ and $\rho = 3/32\pi Gt^2$

$$\Rightarrow d^2\delta/dt^2 + (d\delta/dt)/t - (1/t^2)[1 - 3c_s^2k^2/32\pi G\rho]\delta = 0$$

For $k \rightarrow 0$, the solution $\delta \propto t^n$ with $\delta_+ \propto t^1$ and $\delta_- \propto t^{-1}$

As before, damped oscillations for large k, with a transition near the Jeans Length

How does structure grow?

	above horizon	below horizon	z
radiation dominated epoch	$\delta \propto a^2$ $P \propto a^4$	no growth	
			3500
matter dominated epoch	$\delta \propto a$ $P \propto a^2$	$\delta_{DM} \propto a$, $P_{DM} \propto a^2$, oscillations in baryons	
			1100
after decoupling (if Ω_m~1)	$\delta \propto a$ $P \propto a^2$	$\delta \propto a$ $P \propto a^2$	

Growth of Structure at Early Times

We already examined the evolution for $t >> t_{eq}$ (matter-radiation equality)

But at earlier times a and ρ evolve differently!

Consider now the growth of matter perturbations in a Universe where expansion is driven by a relativistic component.

Assume k = 0 => $d^2\delta/dt^2 + 2((da/dt)/a)d\delta/dt - 4\pi G\rho_m\delta = 0$

If we define $y = \rho_m/\rho_r = a/a_{eq}$ increases with time; y = 1 at $z = z_{eq} \sim 3500$

Then $d^2\delta/dt^2 + 2((da/dt)/a)d\delta/dt - 4\pi G\rho_m\delta = 0$ can be rewritten as

$$\delta'' + (2+3y)\delta'/2y/(1+y) - 3\delta/2y/(1+y) = 0$$

Has 2 solutions: one growing and one decaying. The growing mode:

$$\delta_+ \propto 1+(3/2)y \sim 1 + 5000/(1+z)$$

Before z_{eq}, we have that y < 1 and the growing mode is frozen. This Meszaros effect applies to cold dark matter density fluctuations (not coupled to the radiation via pressure) on large scales.

The total growth from 0 to t_{eq} is $\delta_+(y=1)/\delta_+(y=0) = 5/2$ and afterwards by another factor $1+z_{eq}$

The physical reason for this slow growth is that before t_{eq} the Jeans time is longer than the expansion time. The energy in radiation causes the Universe to expand so fast that the matter has no time to respond.

Growth of Structure

Before decoupling, the dark matter grows normally, i.e., $\delta_{DM} \propto a$

but the baryon dynamics are coupled to that of the radiation.

=> δ_{bary} oscillates like the radiation, so $\delta_{DM} >> \delta_{bar}$

but after decoupling,

$$d^2\delta_{bar}/dt^2 + (4/3t)d\delta_{bar}/dt = 4\pi G(\bar{\rho}_{bar}\,\delta_{bar} + \bar{\rho}_{DM}\,\delta_{DM})$$

$$d^2\delta_{DM}/dt^2 + (4/3t)d\delta_{DM}/dt = 4\pi G(\bar{\rho}_{bar}\,\delta_{bar} + \bar{\rho}_{DM}\,\delta_{DM})$$

If we use that $\delta_m = (\bar{\rho}_{bar}\,\delta_{bar} + \bar{\rho}_{DM}\,\delta_{DM})/(\bar{\rho}_{bar} + \bar{\rho}_{DM}) \sim \delta_{DM}$ and $\Delta \equiv (\delta_{DM} - \delta_{bar})$

$d^2\Delta/dt^2 + (4/3t)d\Delta/dt = 0$ => Δ = constant or $\Delta \propto t^{-1/3}$

$\delta_m \propto t^{2/3} \propto a$

$$\delta_{DM} / \delta_{bar} = (\rho_m\,\delta_m + \rho_{bar}\,\Delta)/(\rho_m\,\delta_m - \rho_{DM}\,\Delta) \rightarrow 1$$

The initial non-zero value of δ_{bar} at decoupling leaves a small effect on δ_m at later times => these are the baryon acoustic oscillations.

Growth of structure

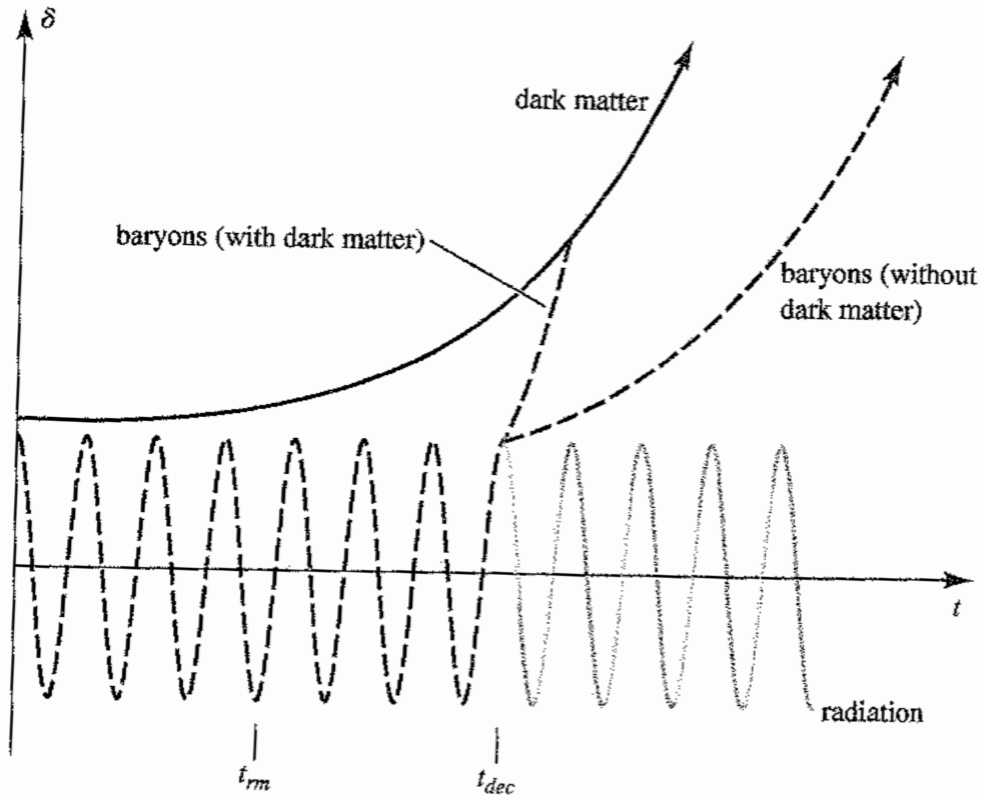
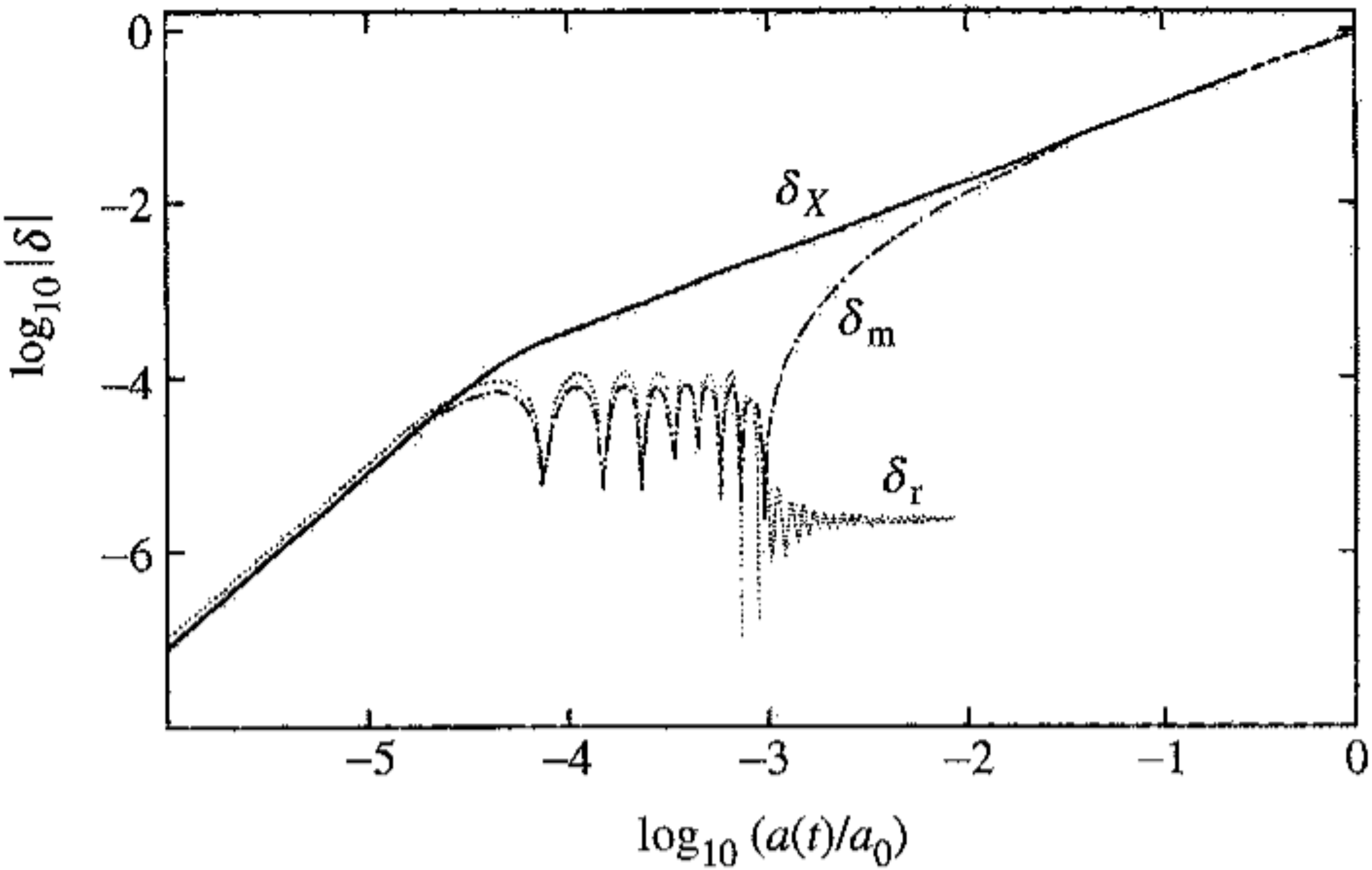


FIGURE 4 A highly schematic drawing of how density fluctuations in different compone-nts of the universe evolve with time.

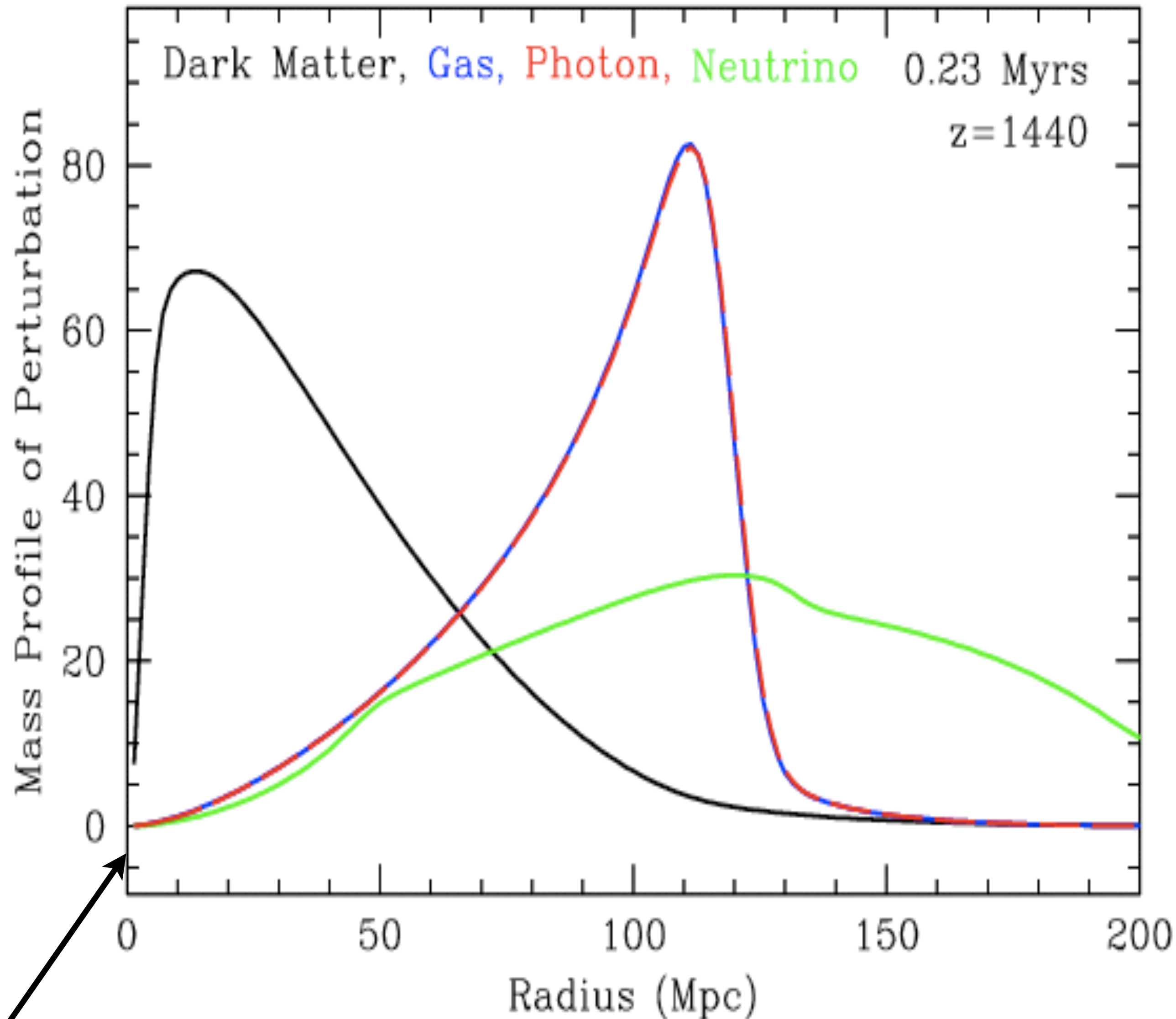
from "Introduction to Cosmology" (Ryden, 2014)

Growth function



From "Cosmology" Coles & Lucchin, 2nd edition

Baryons + DM affect each other after decoupling: How?

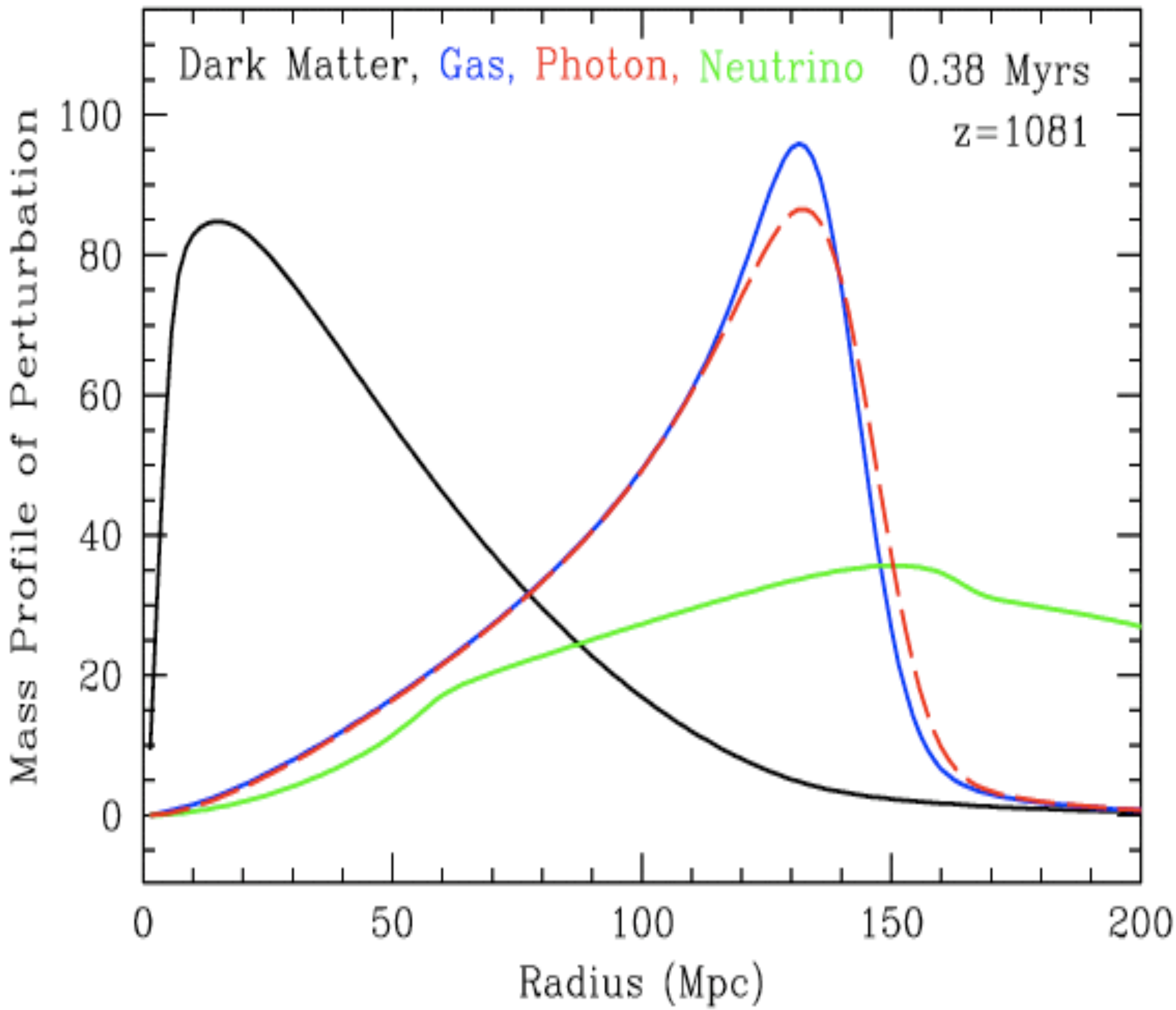


imagine we have a overdensity here at time t = 0

dark matter will fall towards it

but baryons and radiation will bounce

Baryons + DM affect each other after decoupling: How?



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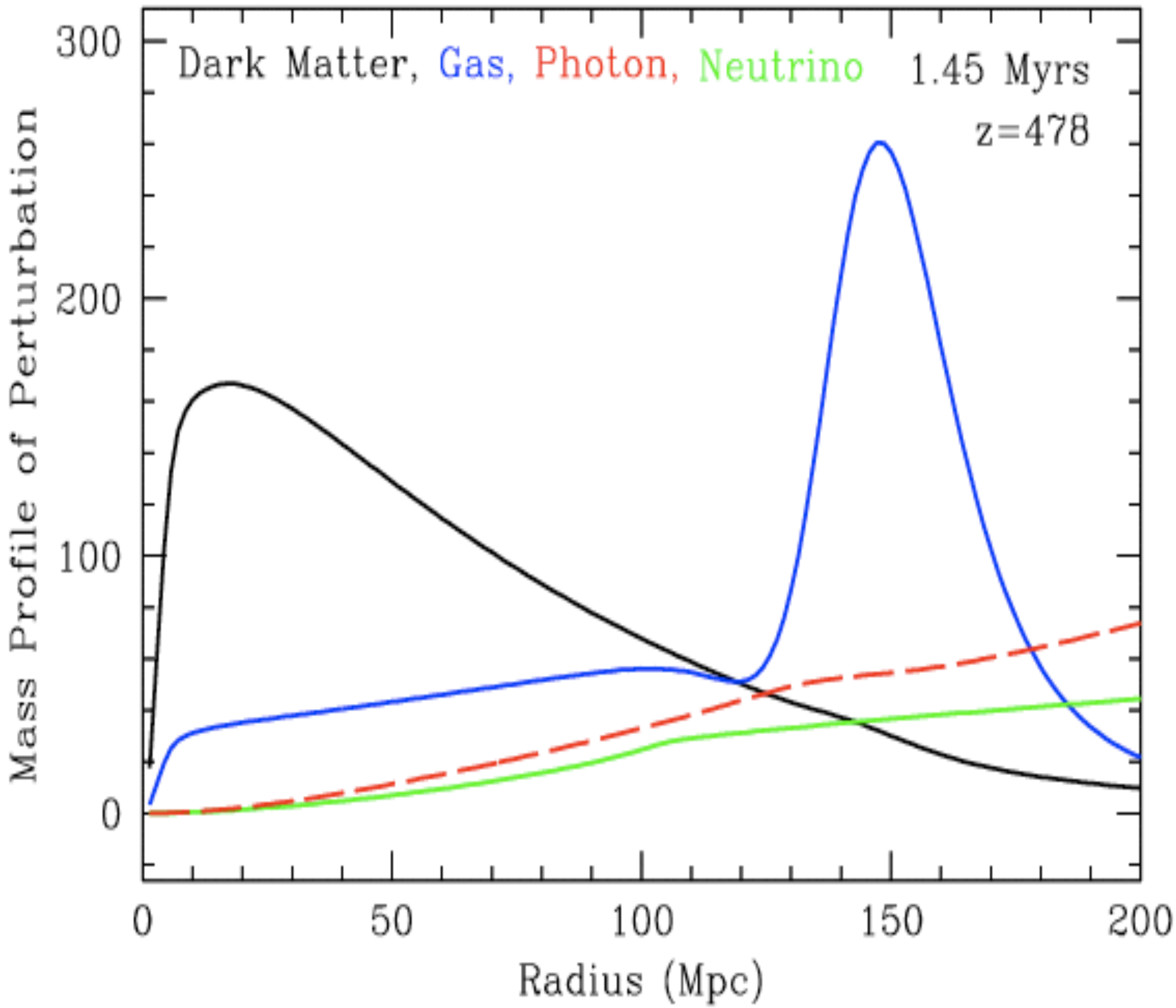

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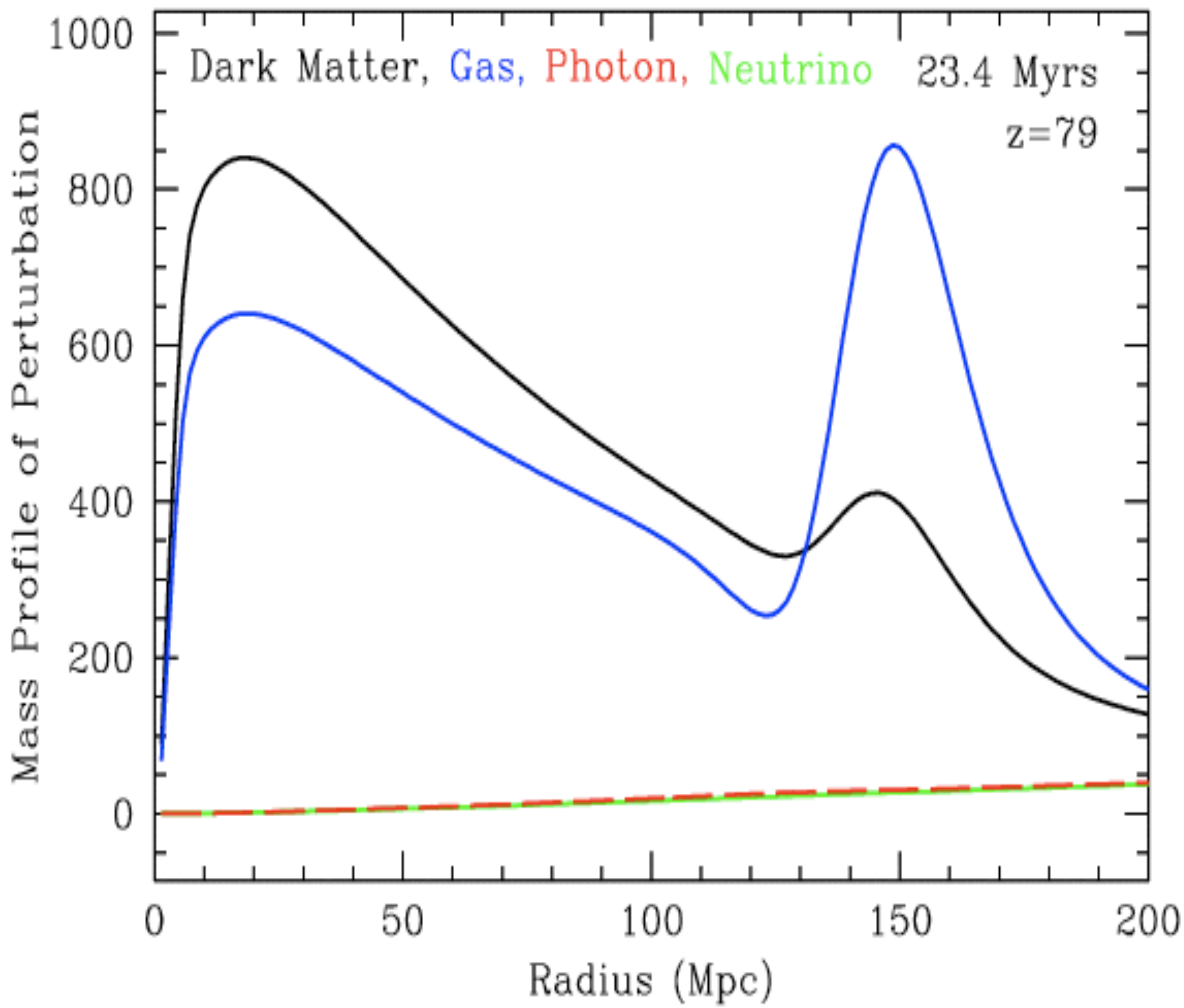
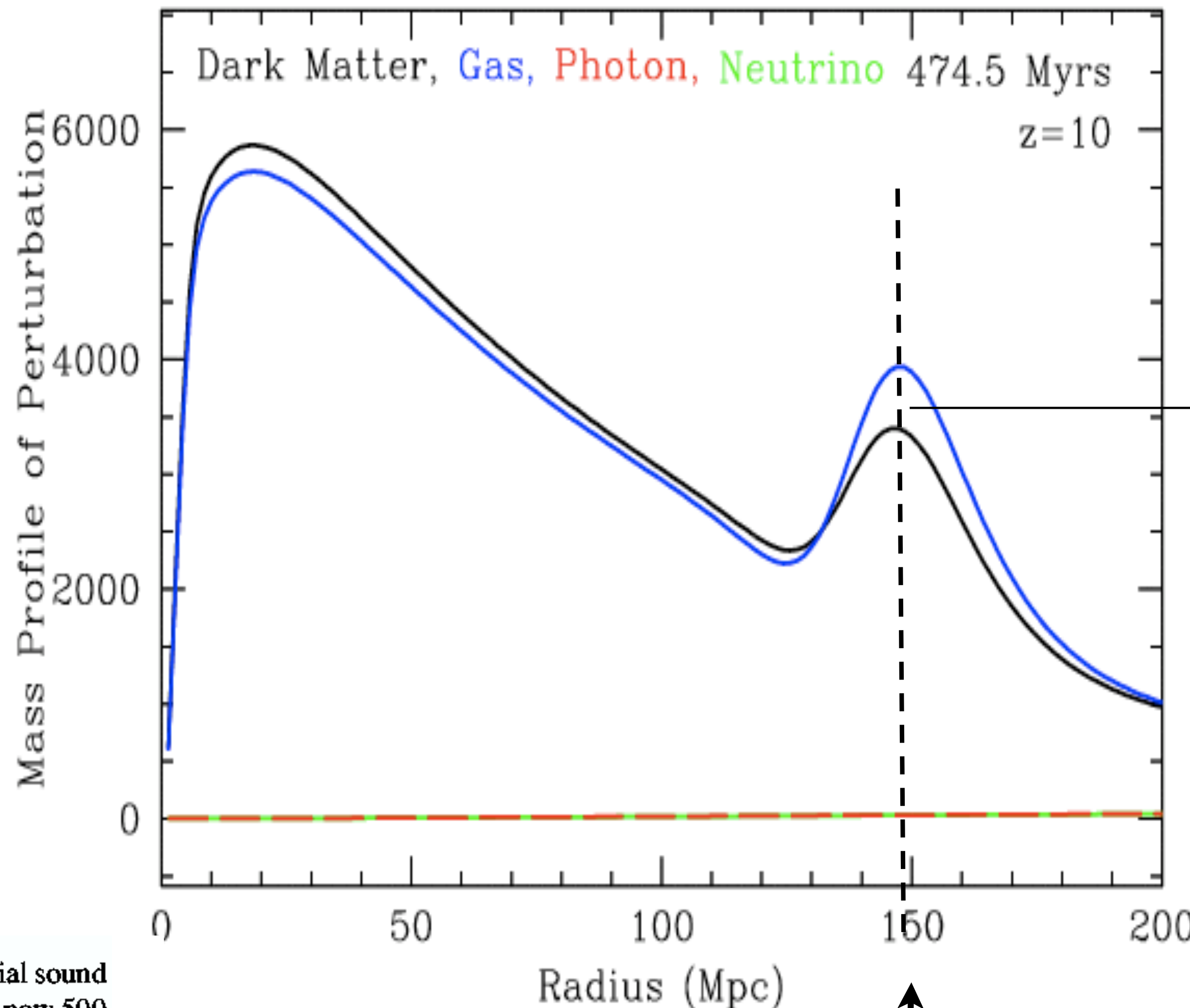
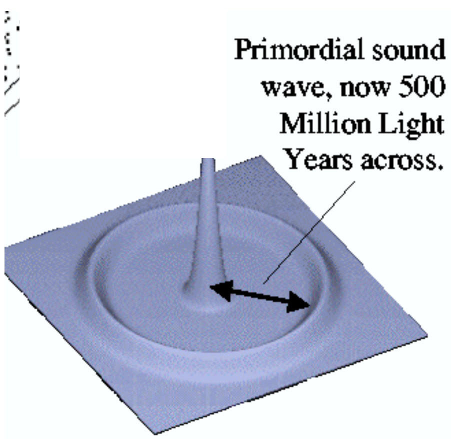


Baryons + DM affect each other after decoupling: How?



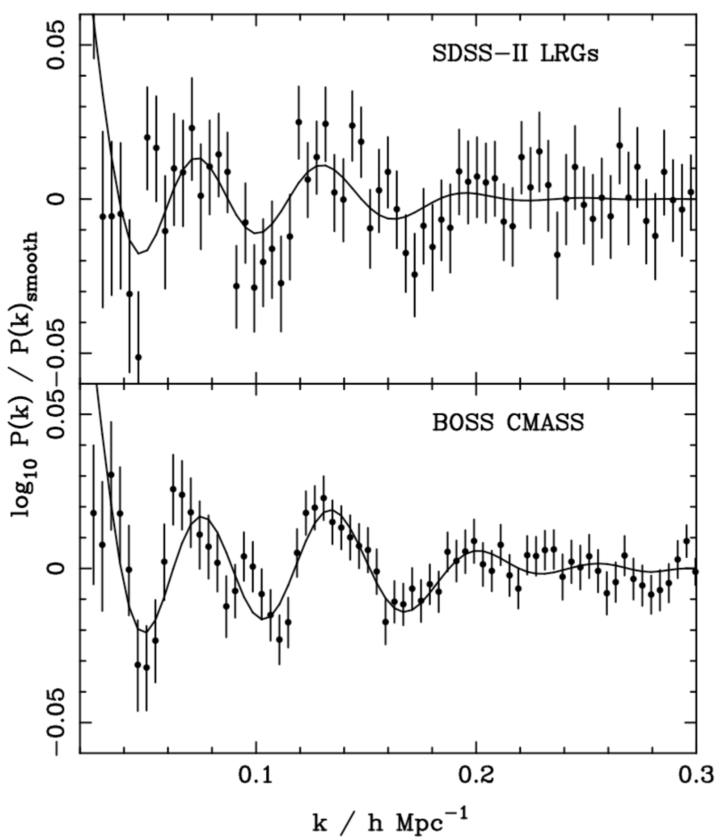
Sound horizon at matter-radiation decoupling

this bump is at 150 Mpc!



Dark matter fluctuations

After decoupling the baryons start following the gravitational potential defined by the dark matter but they retain some of the imprint of the sound-waves at decoupling: the baryon accoustic oscillations.



Anderson et al. (2012)

New Material for This Week

Power Spectrum

If the overdensities/underdensities δ are a Gaussian field (as predicted by many theories) then the power spectrum P(k) completely specifies the statistical properties.

P(k) quantifies the amount of clustering for each k-mode.

The power spectrum is defined as

$$\langle \delta(k)\delta^*(k') \rangle = (2\pi)^3\, \delta_D(k-k')P(k)$$

isotropy implies that P(k) can only depend on |k|

where δ_D is the delta Dirac function.

Measuring the Matter Power Spectrum From Galaxies: Correlation Function

The two-point correlation function gives the excess probability of finding pairs of objects at a separation r. It is defined as $\xi(r) = <\delta(x_1)\delta(x_2)>$ and this is related to the power spectrum through its Fourier Transform

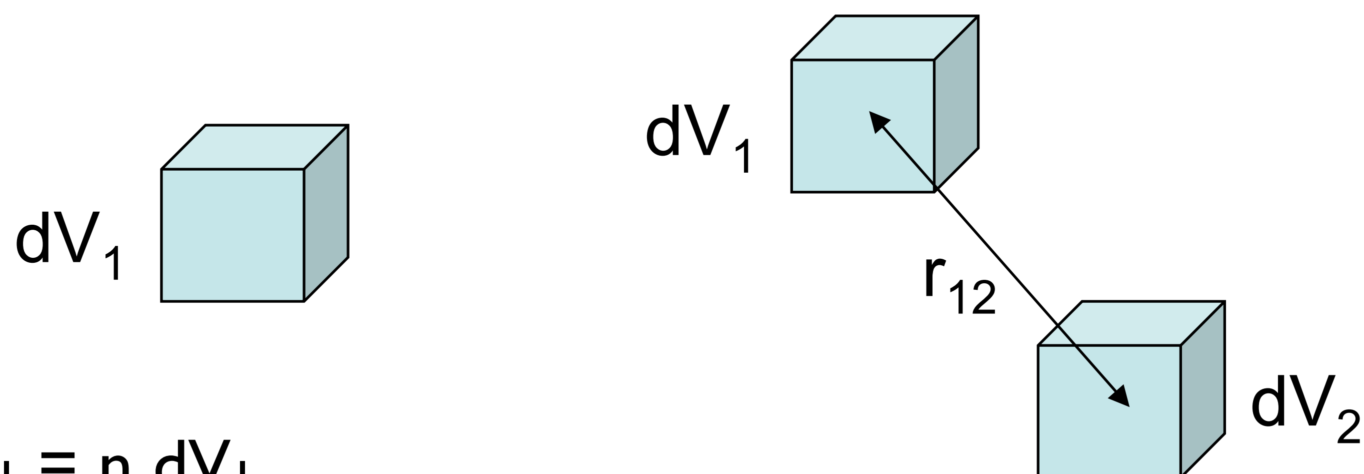
$$\xi(r) = <\delta(x_1)\delta(x_2)> = \int d^3k/(2\pi)^3 e^{ik \cdot x} P(k)$$

The power spectrum has units of length and it is convenient to define a dimensionless version:

$$\Delta^2(k) = (4\pi k^3 P(k))/(2\pi)^3$$

We quantify clustering in terms of correlation functions

The Correlation function ξ is not equal to zero -- since the presence of a galaxy at some place in space makes it more likely another one will be close by....

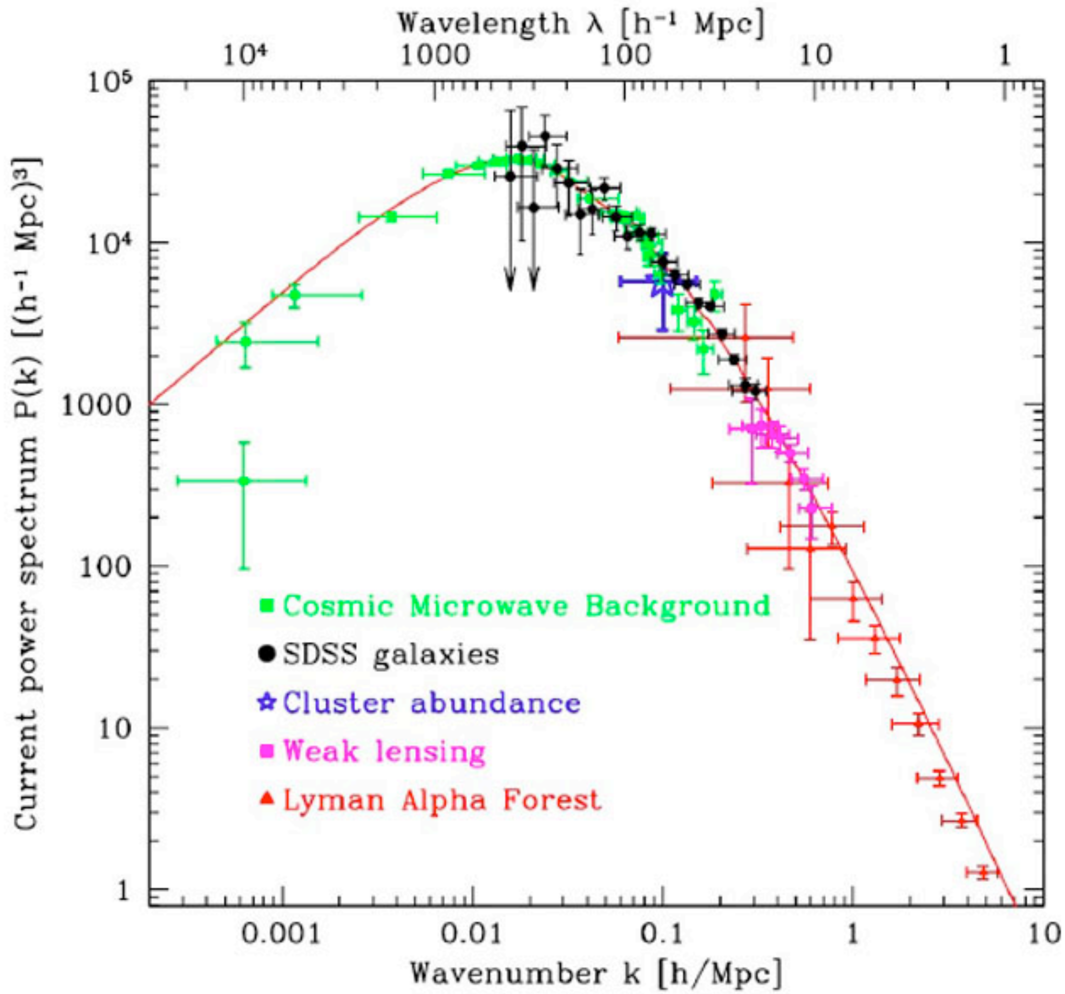


$$dP_1 = n\, dV_1$$

$$dP_{12} = n^2\,(1 + \xi(r_{12}))\, dV_1\, dV_2$$

n = average density of galaxies

Statistics of matter fluctuation



From Tegmark (2004)

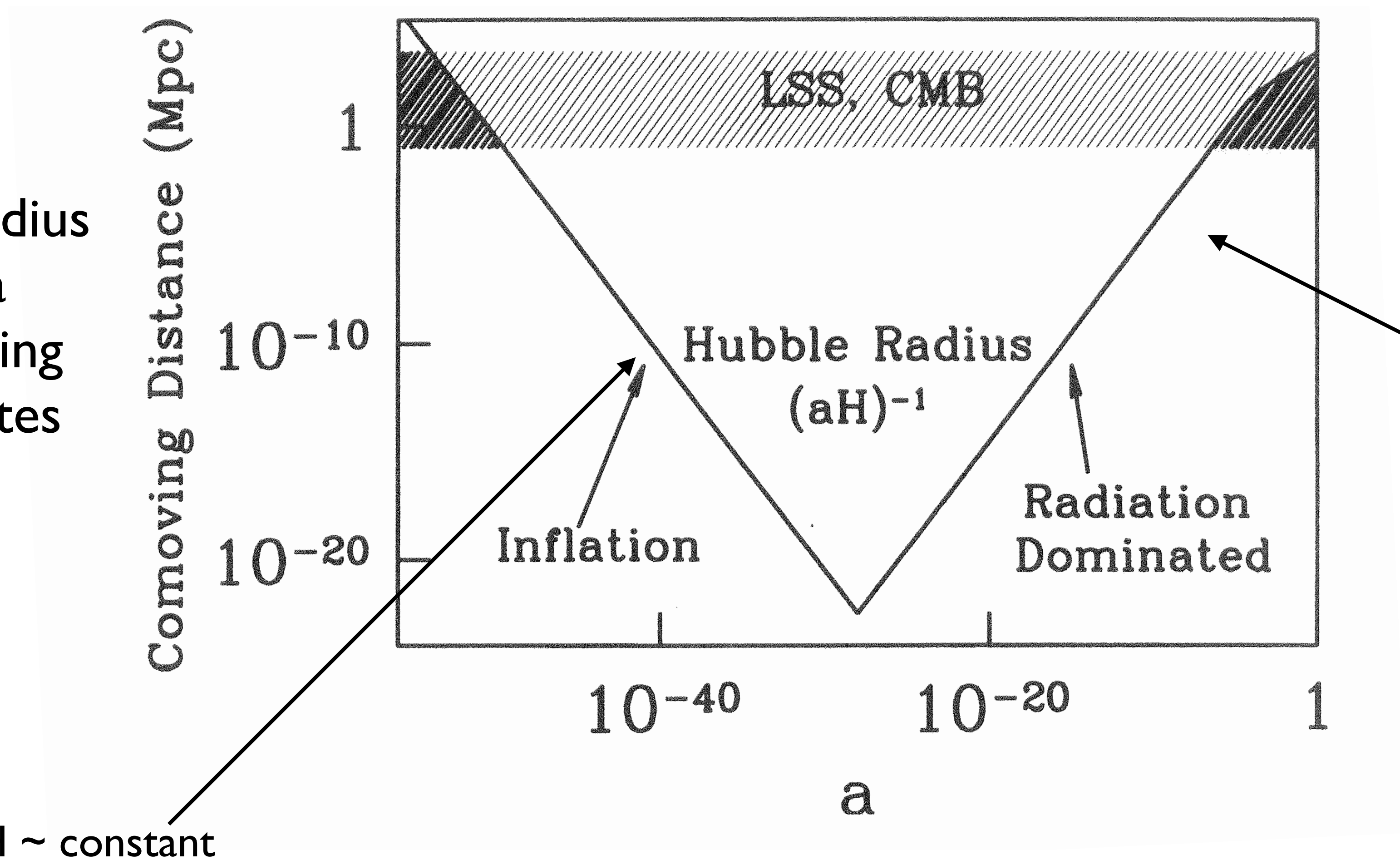
What is the origin of fluctuations in the energy density in the first place?

What is the origin of fluctuations in the energy density in the first place?

Remember the situation regarding Horizons and Inflation:

Inflation

Hubble Radius
(c/H)/a
in comoving
coordinates

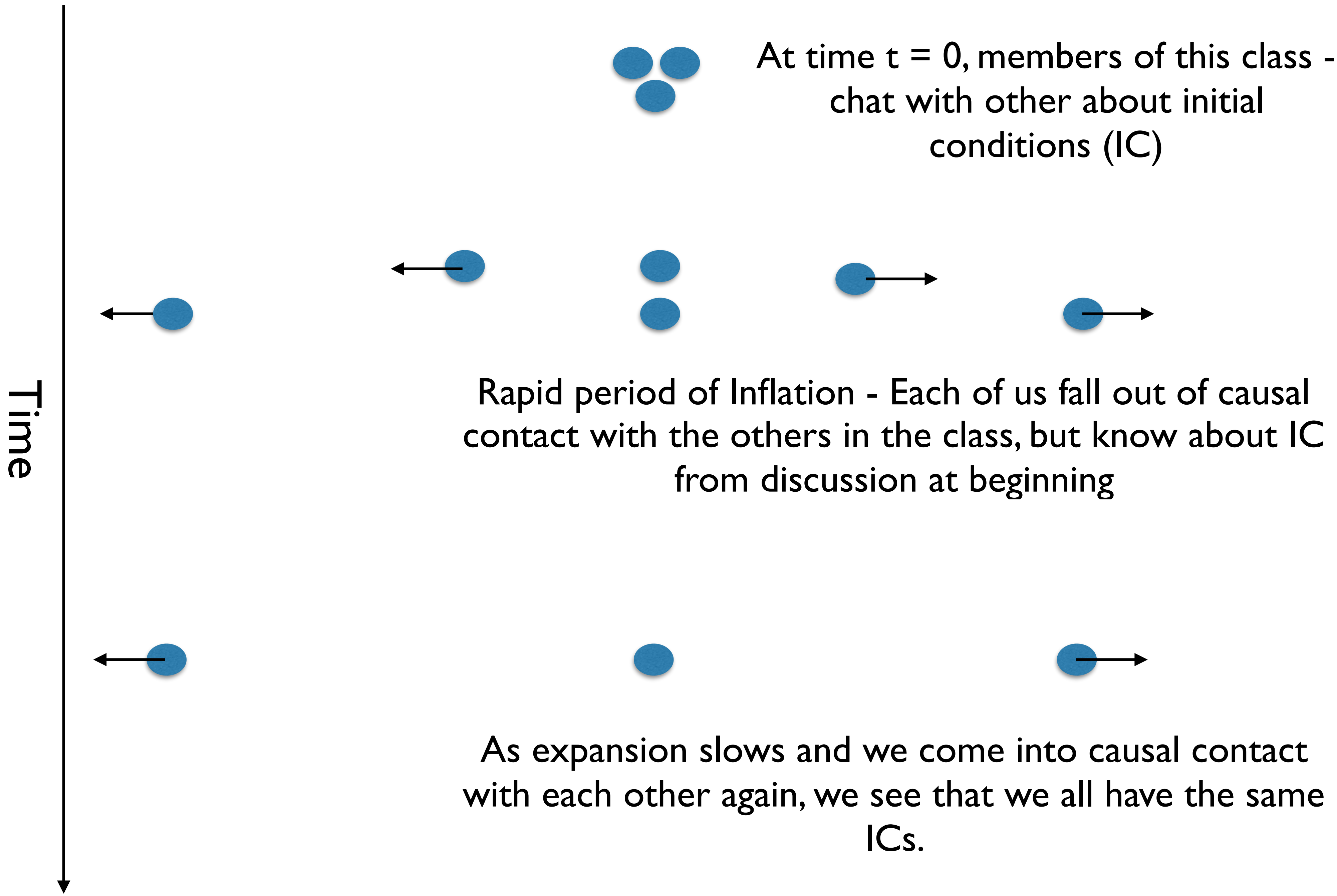


scales as a

H ~ constant
scales as 1/a
$a \propto e^{(\Lambda/3)t}$

Dodelson (2003)

Let's take as an apology the following situation:



Setting Up Primordial Power Spectrum From Inflation

Under non-expanding circumstances, quantum fluctuations die out quickly, but during inflation the expansion is so fast that any fluctuation is moved outside the horizon of any compensating fluctuation. By the time they are back in each other horizon they are in back in each other's horizon, they are no longer quantum scale:

$$a(t) = a(t_{infl})\, e^{H(t - t_{infl})}$$

← when inflation started

How long does it take for a quantum fluctuation of size λ_{quant} to freeze out? i.e., what is $\Delta t \sim t - t_{infl}$?

It is given by the time it takes for the fluctuation to expand to the Hubble radius

$$(a_{freeze}/a_{quant})\lambda_{quant} = r_H = c/H$$

$$\Rightarrow \Delta t = (1/H)\ln(a_{freeze}/a_{quant}) = (1/H)\ln(c/H\lambda_{quant})$$

During inflation, $H \sim$ constant and we can reasonably assume the same for λ_{quant}, Δt is constant.

If during inflation, perturbations are generated at a given rate => fixed number per logarithmic interval in space (because of exponential expansion). This continues for many e-folding times and during each interval, the fluctuation looks the same => i.e. power spectrum must be scale free => power law $P(k) \propto k^n$

Setting Up Primordial Power Spectrum From Inflation

Let us take the primordial power spectrum $P(k)$ to have a form Ak^n;

If the scalar field that is perturbed is related to the gravitational potential Φ and the fluctuations are of the same amplitude => Δ_Φ^2 = constant

i.e., fluctuations are scale-invariant in the gravitational potential Φ

$\Delta_\Phi^2 \equiv k^3 P_\Phi(k) \propto$ constant   but $\nabla^2\Phi = 4\pi G\rho\delta$ => $k^2\Phi \propto -\delta(k)$

Fourier Transform

$k^4 P_\Phi(k) \propto P_\delta(k)$ using the definition of $P(k)$

$P_\delta(k) \propto k^4 P_\Phi(k) \propto k(k^3 P_\Phi(k)) \propto k\Delta_\Phi^2 \propto k$

=> $n=1$ assuming $P_\delta(k) \propto k^n$

a $n=1$ model is called a Harrison-Zeldovich spectrum

How does structure grow?

	above horizon	below horizon	z
radiation dominated epoch	$\delta \propto a^2$ $P \propto a^4$	no growth	
			3500
matter dominated epoch	$\delta \propto a$ $P \propto a^2$	$\delta_{DM} \propto a$, $P_{DM} \propto a^2$, oscillations in baryons	
			1100
after decoupling (if $\Omega_m \sim 1$)	$\delta \propto a$ $P \propto a^2$	$\delta \propto a$ $P \propto a^2$	

Different Growth of Structure on Small + Large Scales

The observed power spectrum is quite different than the primordial power spectrum

$P(k) = Ak^n T^2(k)$

Transfer function

T(k) captures the growth of fluctuations in and outside the horizon.

Inflation sends perturbations beyond the horizon, but after the end of inflation the horizon is expanding again.

Perturbations that have not yet entered the horizon continue to grow (to demonstrate this requires a rigorous GR treatment); we saw that $\delta(k)$ grows as a^2
=> P(k) grows as a^4

Large scale modes enter later and thus have had more time to grow, but if a mode enters the horizon during radiation domination its growth will cease and instead oscillate due to the radiation pressure.

This does not apply to the DM: during radiation domination it is the radiation fluid that produces the growth of modes, but the DM interacts only through gravity. The density of DM can stream into the gravitational well produced by a perturbation in the radiation fluid, but on small scales this averages out.

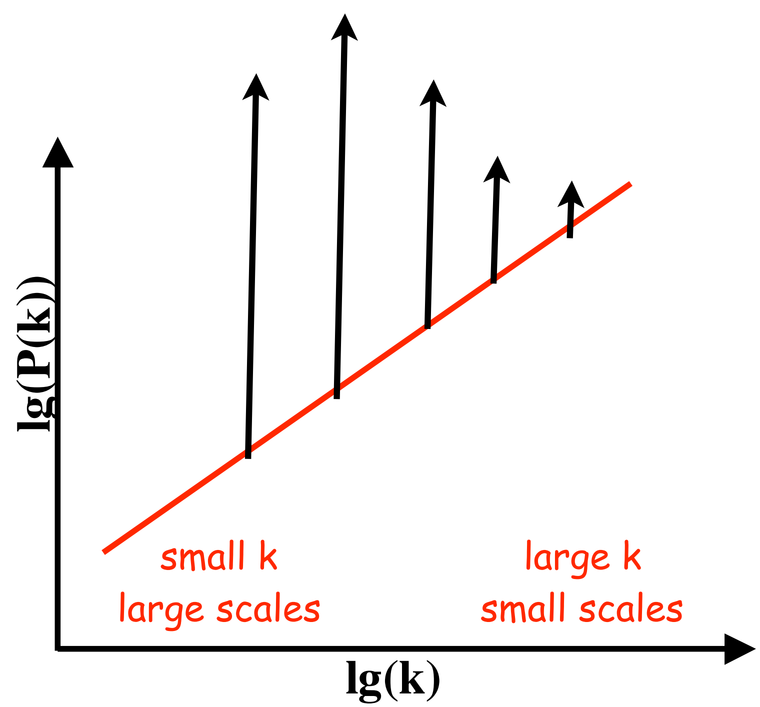
DM growth essentially stalls.

Illustrating why there is a peak in the matter power spectrum

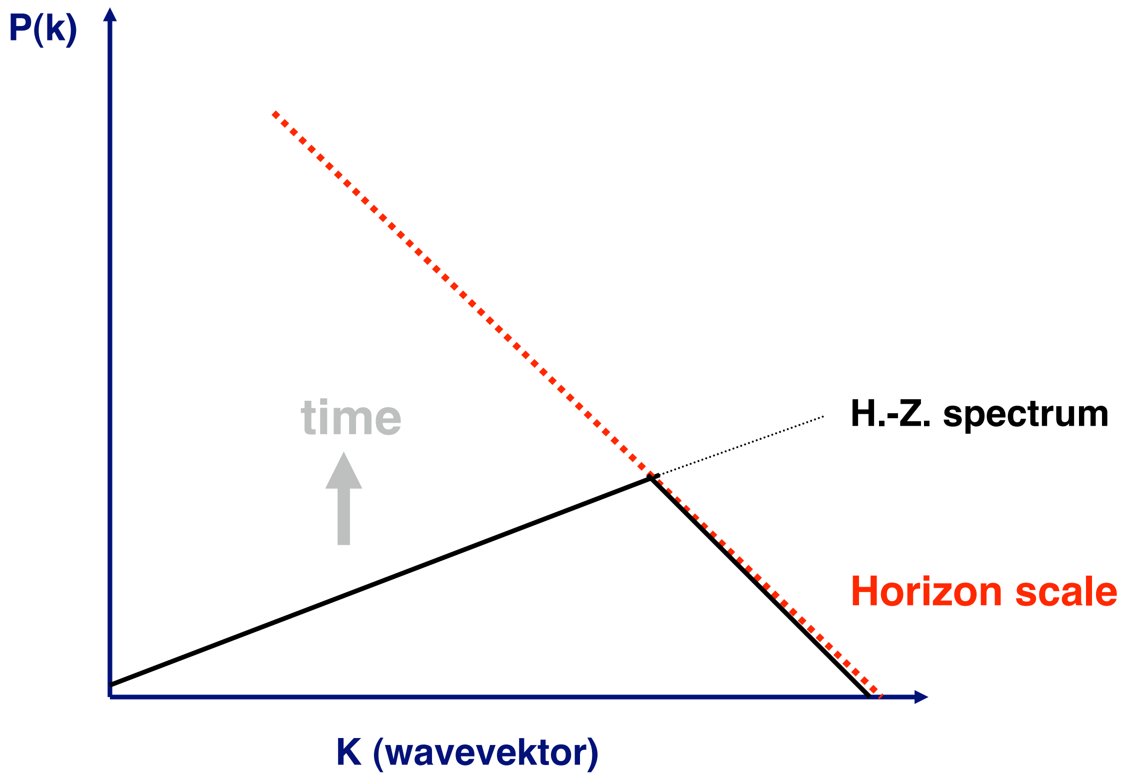
The initial power spectrum of fluctuations is the following:

$$P_0(k) = A\, k^{n_s}$$

Therefore we could expect P(k) at large scales to grow much more than at small scales



Evolution of the Matter Power Spectrum

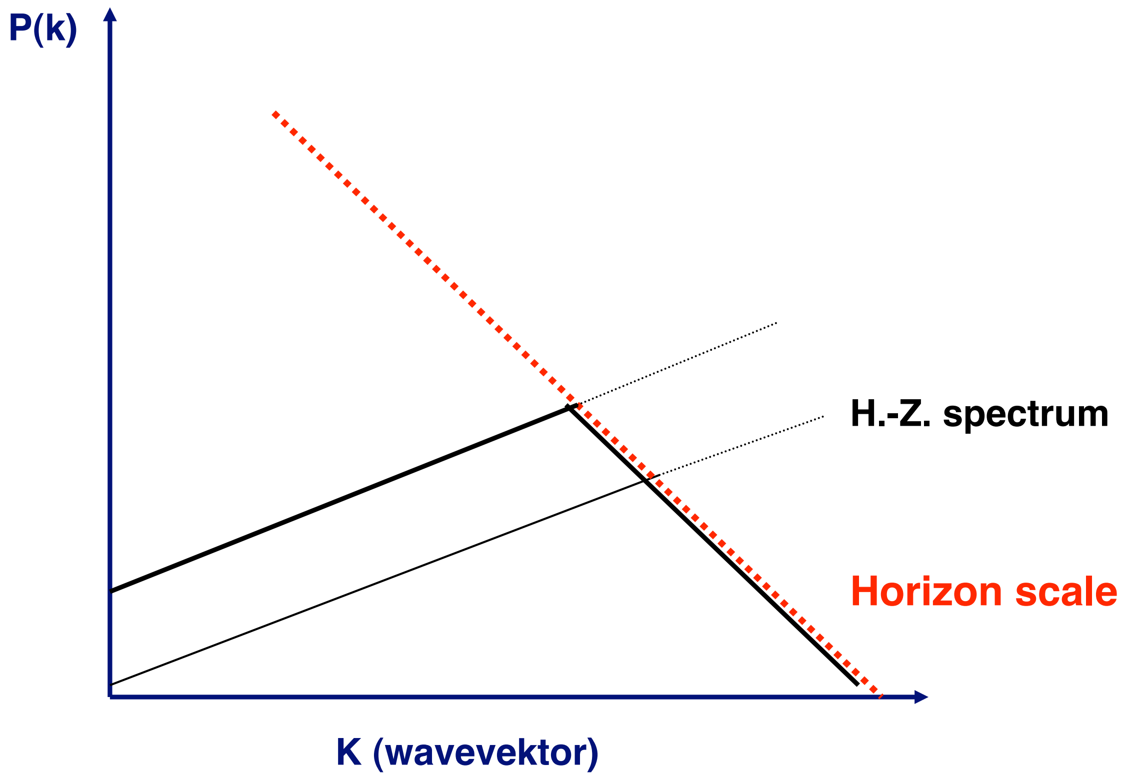


large scales
small k

small scales
large k

Credit: Bohringer

Evolution of the Matter Power Spectrum



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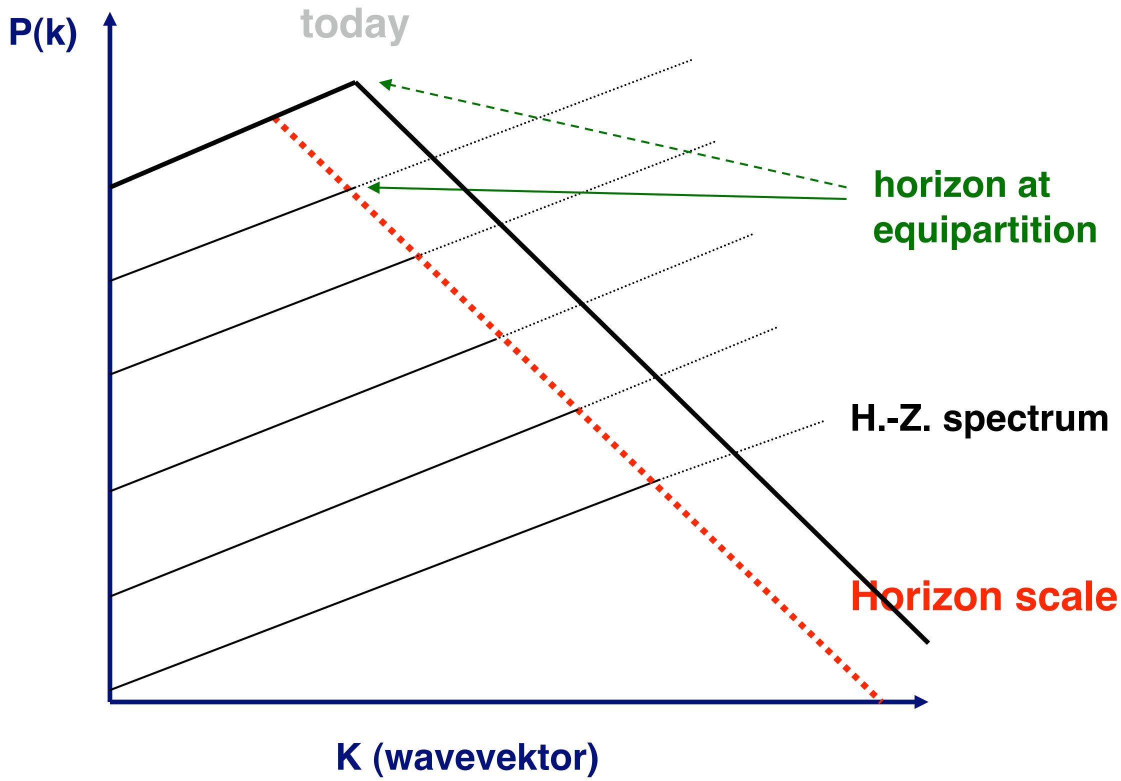
Evolution of the Matter Power Spectrum



H. Böhringer

large scales
small k

small scales
large k

Credit: Bohringer

Setting Up Primordial Power Spectrum From Inflation

What is the power spectrum which results?   Let's quantify issues:

Modes entering after a_{eq} have $P(k) \propto k$ and have grown by $(a_{eq}/a_i)^4$

$P_{eq}(k) = (a_{eq}/a_i)^4 P_i(k) \quad k << k_{enter\text{-}eq}$

Modes that enter before a_{eq} grew by factor $(a_{enter}(k)/a_i)^4$ where $a_{enter}(k) \propto 1/k$

$\Rightarrow P_{eq}(k) \propto k^{-3}$ for $k >> k_{enter\text{-}eq}$

The power spectrum peak around $k \sim k_{enter\text{-}eq}$

Where is the peak of the power spectrum?

The position of the peak of the power spectrum depends on the Horizon Size

$d_H = 2c/H = 2c/H_0 (\Omega_{m,0})^{0.5} (1+z)^{-1.5}$

which is equal to the above in a matter-dominated universe at $z >> 1/\Omega_{m,0}$

A feature of some some length l grows in proportion to a, but as the horizon grows as $d_H \propto a^{3/2}$, so larger features come into causal contact with each other at later times

=> we therefore expect the transfer function to depend on $\Omega_m h^2$ and k

Where is the peak in the power spectrum?

We already found that for $y \equiv \rho_m/\rho_r = a/a_{eq}$ that $\delta_m \propto 1 + (3/2)y$

=> $\delta_m \propto$ constant for $a << a_{eq}$ and $\delta_m \propto a$ for $a >> a_{eq}$

In the radiation-dominated era perturbation modes with $l < d_H(z_{eq})$ enter the horizon but δ is constant.

In the matter-dominated era, all modes grow and $\delta \propto a$

=> the power spectrum must have a break on the length scale of the horizon at matter radiation equality:

$d_H(z_{eq}) \sim 16/(\Omega_{m,0}h^2)$ Mpc $\qquad\qquad k \sim 0.06\ \Omega_{m,0}h^2\ \mathrm{Mpc}^{-1}$

For $k > k_{eq}$ (small scales) fluctuations enter the horizon during the radiation dominated era and cannot grow => $P(k) \propto k^{-3}$

For $k < k_{eq}$ (large scales) fluctuations enter the horizon during the matter dominated era and grow, i.e., preserving the initial power spectrum $P(k) \propto k$

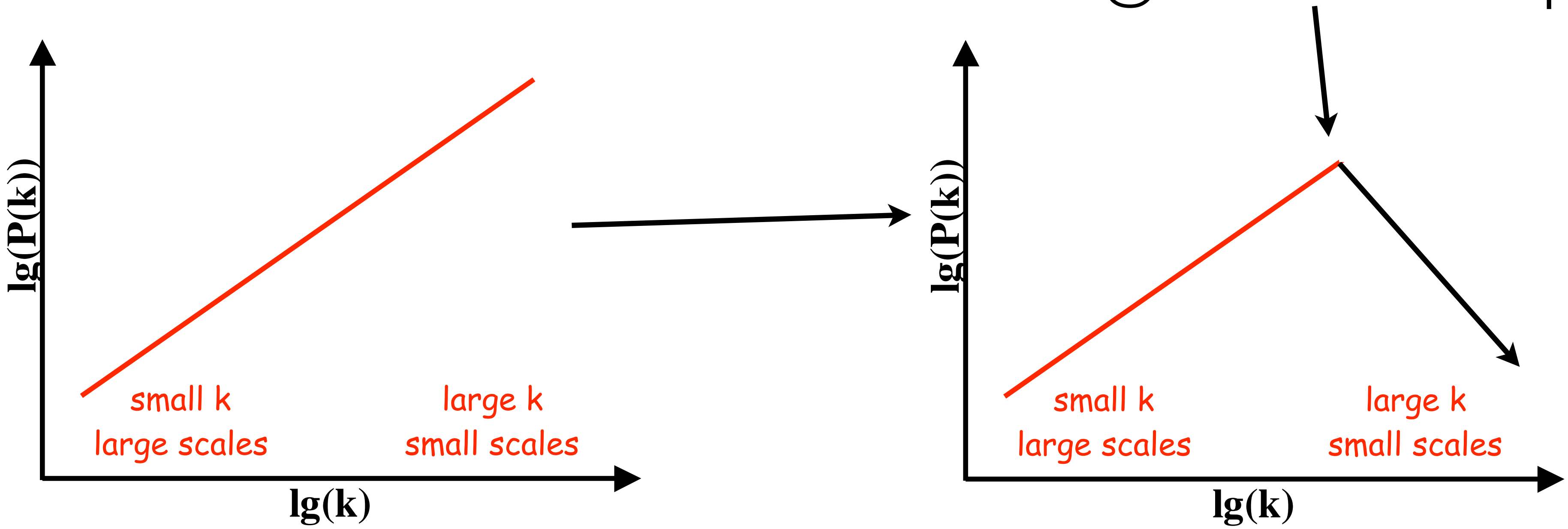
The baryon acoustic oscillations are superimposed on the dark-matter fluctuations.

Illustrating why there is a peak in the matter power spectrum

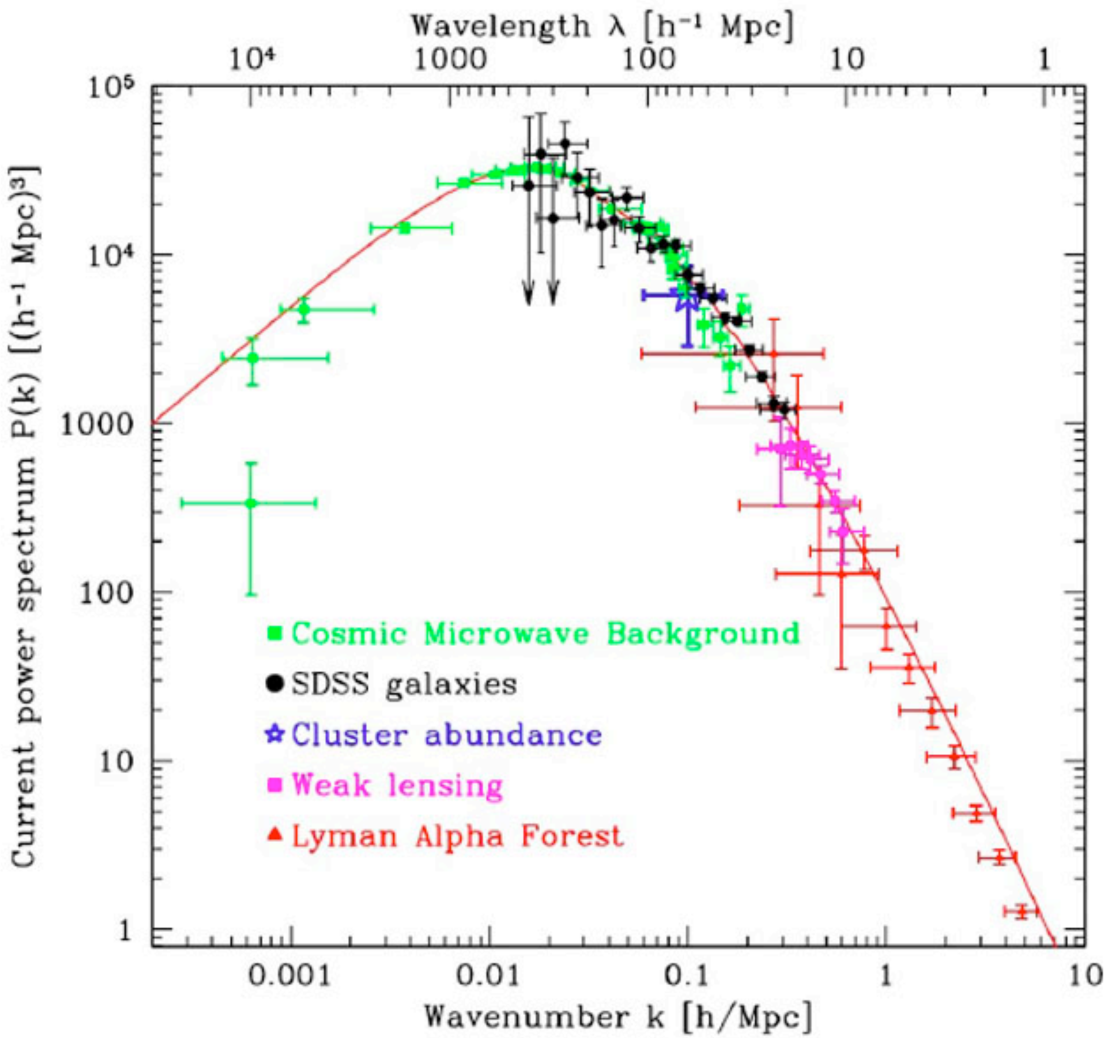
The initial power spectrum of fluctuations is the following:

$$P_0(k) = A\, k^{n_s}$$

Position of turn-over determined by horizon size @ matter-radiation equality



Statistics of matter fluctuation



From Tegmark (2004)

Transfer functions

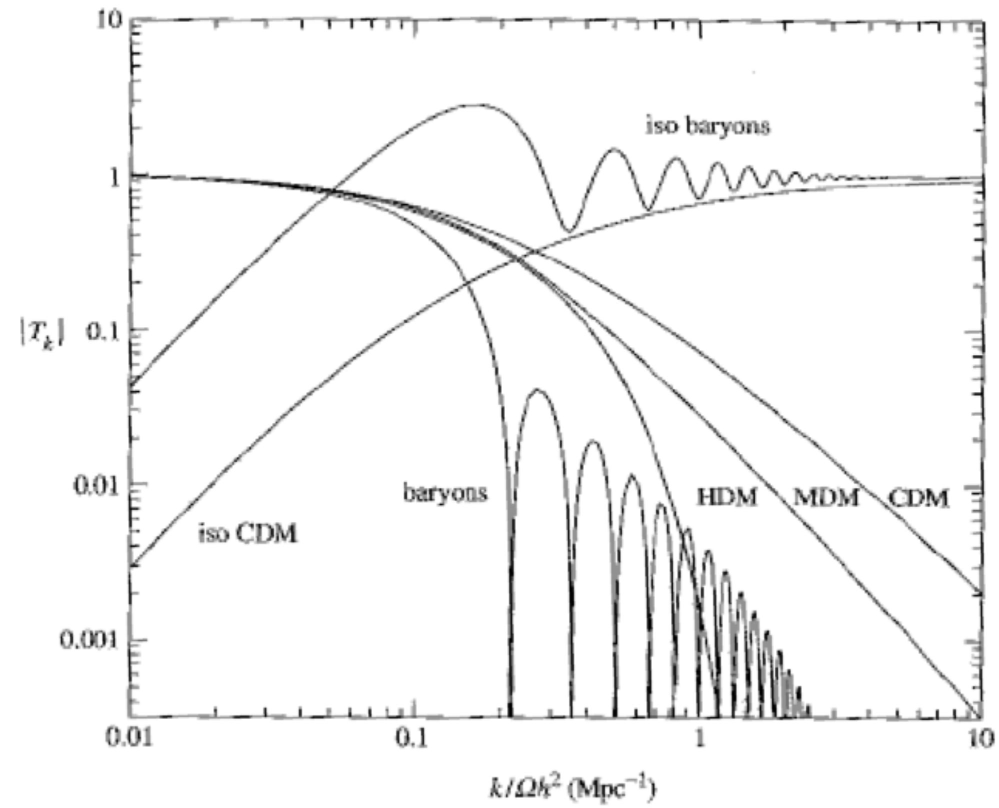
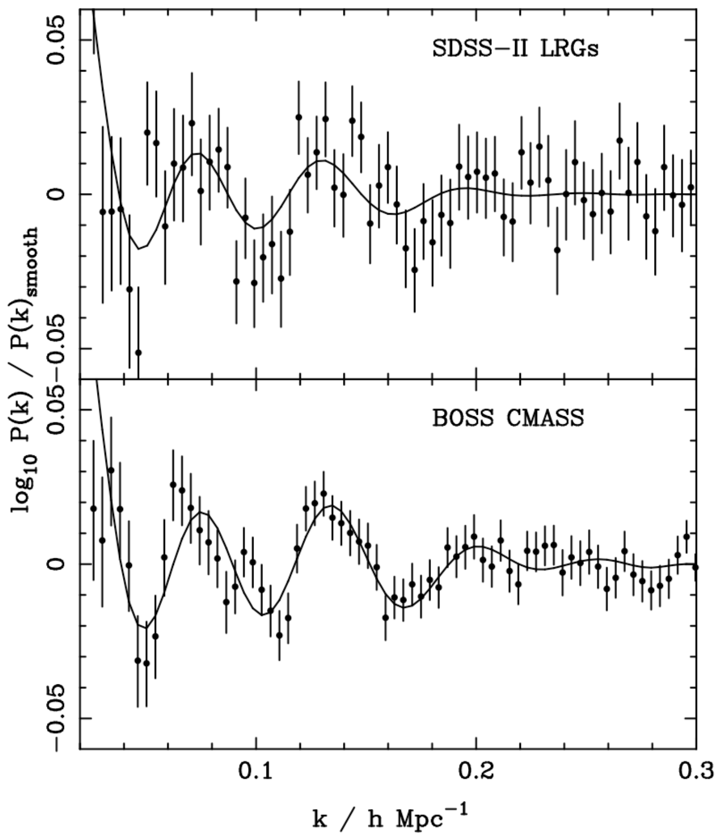


Figure 15.1 Examples of adiabatic transfer functions for baryons, hot dark matter (HDM), cold dark matter (CDM) and mixed dark matter (MDM; also known as CHDM). Isocurvature modes are also shown. Picture courtesy of John Peacock.

Dark matter fluctuations

After decoupling the baryons start following the gravitational potential defined by the dark matter but they retain some of the imprint of the sound-waves at decoupling: the baryon accoustic oscillations.



Anderson et al. (2012)

Adiabatic Perturbations

In what form do fluctuations in the radiation/matter energy density take? How large are fluctuations in matter relative to fluctuations in radiation and also in the temperature?

How does δ_m compare to δ_r ?

Before recombination, the baryons and the radiation were tightly coupled. The entropy per unit mass in a volume has a very high value because of the large value of σ_{rad} (entropy per baryon).

We discussed earlier that the value of σ_{rad} might be related to the microscopic physics of a GUT or electroweak phase transition; if that is correct, then we expect fluctuations to have the same value for σ_{rad} => we expect adiabatic perturbations.

=> entropy is carried almost entirely by radiation

$$S = (4/3)\sigma T^3 V \propto \sigma_{rad} \propto T^3/\rho_m \propto \rho_r^{3/4}/\rho_m \qquad \sigma_{rad} = 4 m_p \sigma_r T^3 / 3 k_B \rho_m$$

An adiabatic perturbation leaves S invariant and consists of fluctuations in both ρ_m and ρ_r such that

$$\delta S/S = 0 = \delta\sigma_{rad}/\sigma_{rad} = 3\ \delta T/T - \delta\rho_m/\rho_m = (3/4)\ \delta\rho_r/\rho_r - \delta\rho_m/\rho_m$$

$$\delta_m \equiv \delta\rho_m/\rho_m = 3\ \delta T/T = (3/4)\ \delta\rho_r/\rho_r \equiv 3/4\ \delta_r$$

Exam will take place 10-13 PM
on December 21

Have you registered?

What will be on the exam?

1. Exam will cover very similar material to the problems on the problem sets. Expect one problem on the exam very similar to one of these. One problem may be more or less identical to a homework problem.

2. Be familiar with all the basic derivations done in class and major concepts: Friedman's equations, fluid equations, particle horizon, freeze out, equation of state parameters, adiabatic expansion and entropy conservation.

3. Have a thorough understanding of the challenges with the Big Bang theory and be capable of explaining / quantifying how inflation can address those challenges.

4. Know especially the material from the lecture that is quickly reviewed at the beginning of the following lecture, as that is the most important.

What will be on the exam?

5. You should be familiar with all the material discussed in those sections of the Barbara Ryden book covered in class and a few parts of Coles & Lucchin. You might expect exam problems on the level of the material in that book.

6. You will not be expected to memorize every equation covered in class, but if equations come up repeatedly in different lectures, I would strongly suggest being familiar with them and their dependencies on various variables.

7. You should expect ~4-5 problems with multiple parts on the exam. There will be a number of short answer questions where I ask you to describe various important concepts covered in class.