Sterrewacht Leiden

Origins & Evolution of the Universe

an introduction to cosmology — Fall 2018

Lecture 11: Introduction to Structure Formation (II)

Rychard Bouwens

Layout of the Course

Sep 24: Introduction and Friedmann Equations
Oct 1: Fluid and Acceleration Equations
Oct 8: Introductory GR, Space Time Metric, Proper Distance
Oct 15: Redshift, Horizons, Observable Distances
Oct 17: Problem Class #1
Oct 22: Observable Distances, Parameter Constraints
Oct 29: Thermal History, Early Universe
Nov 5: Early Universe, Inflation
Nov 12: Inflation, Lepton Era, Big Bang Nucleosynthesis
Nov 14: Problem Class #2
Nov 19: Recombination, Cosmic Microwave Background Radiation
Nov 26: CMB Radiation (II), Introduction to Structure Formation
Dec 3: Introduction to Structure Formation (II)
Dec 5: Problem Class #3
Dec 10: Finishing Thoughts, Review
Dec 21: Final Exam

Problem set #3 was mailed to you 1.5 weeks ago

Due this Wednesday 13:30 December 5

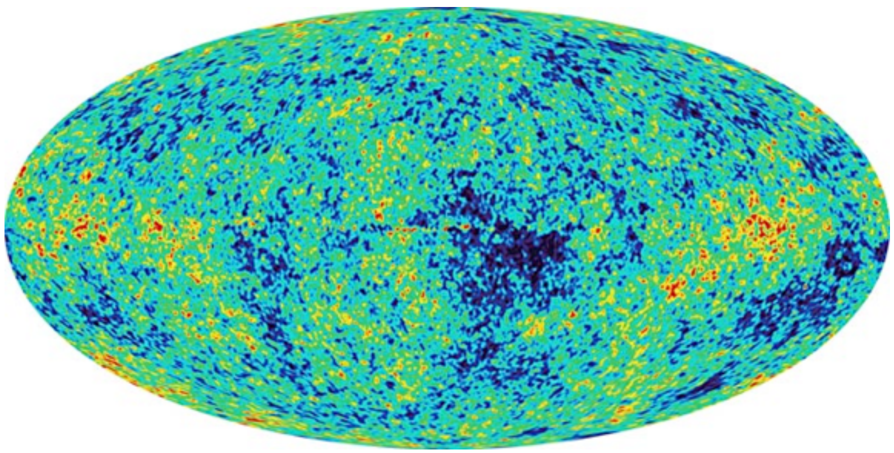
Exam will take place 10-13 PM
on December 21

Have you registered?

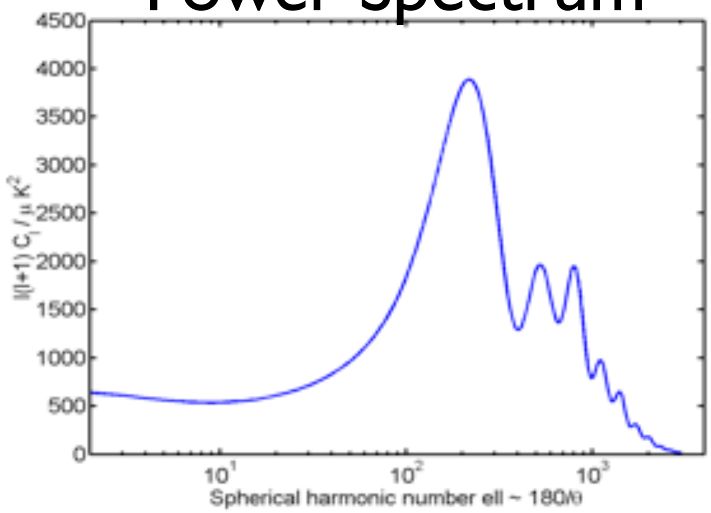
Review Last Week

Power Spectra Derived from Fluctuations in CMB

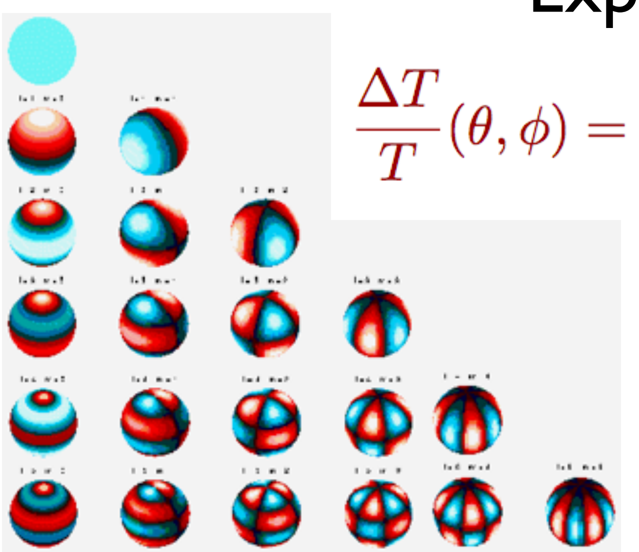
-- Use the spherical harmonic expansion to construct a power spectrum to describe anisotropies of the CMB on the sky



Power Spectrum



Expansion:



$$\frac{\Delta T}{T}(\theta,\phi) = \sum_{\ell=1}^{\infty} \sum_{m=-\ell}^{\ell} a_{\ell m} Y_m^{\ell}(\theta,\phi)$$

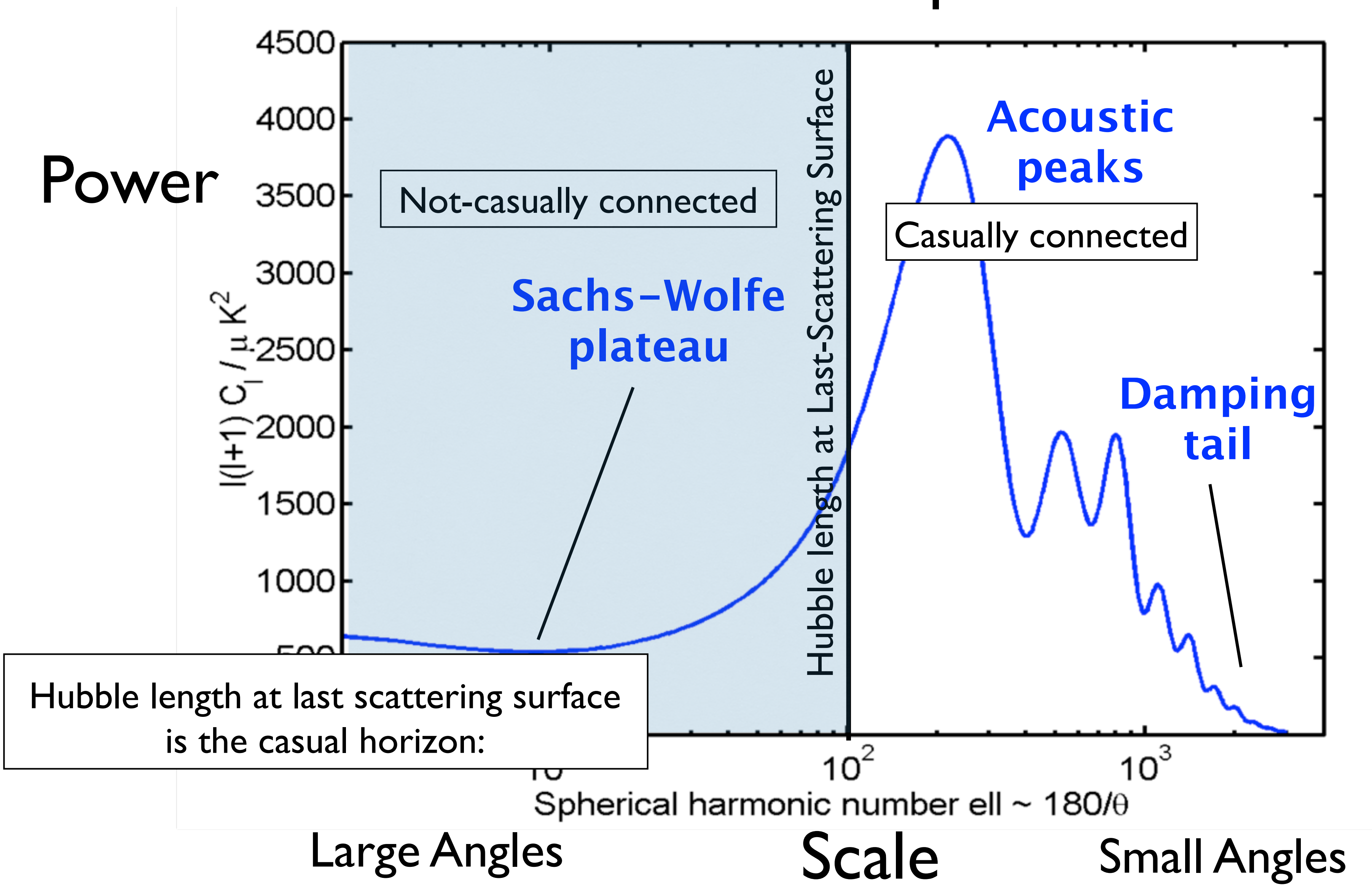
l = 180 / θ

After deriving the a_{lm} coefficients from the data, determine the statistical average

$$c_\ell = \langle |a_{\ell m}| \rangle^2$$

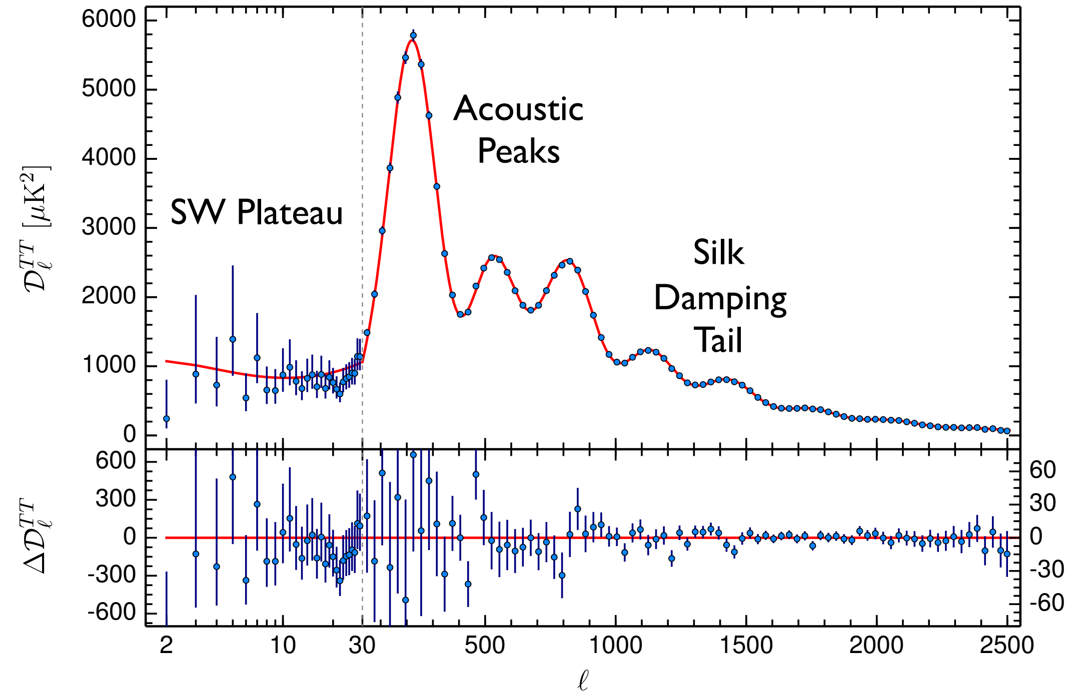
First question: how large can the angle become before the regions become casually disconnected?

Here is such a spectrum:



Cosmic Microwave Background

TT spectrum



Sachs-Wolfe Plateau: Constrain normalization of primordial power spectrum

1st acoustic peak: Measure Angular Diameter Distance to Last Scattering Surface

Ratio of Even and Odd Acoustic Peaks: Probe Baryon Content

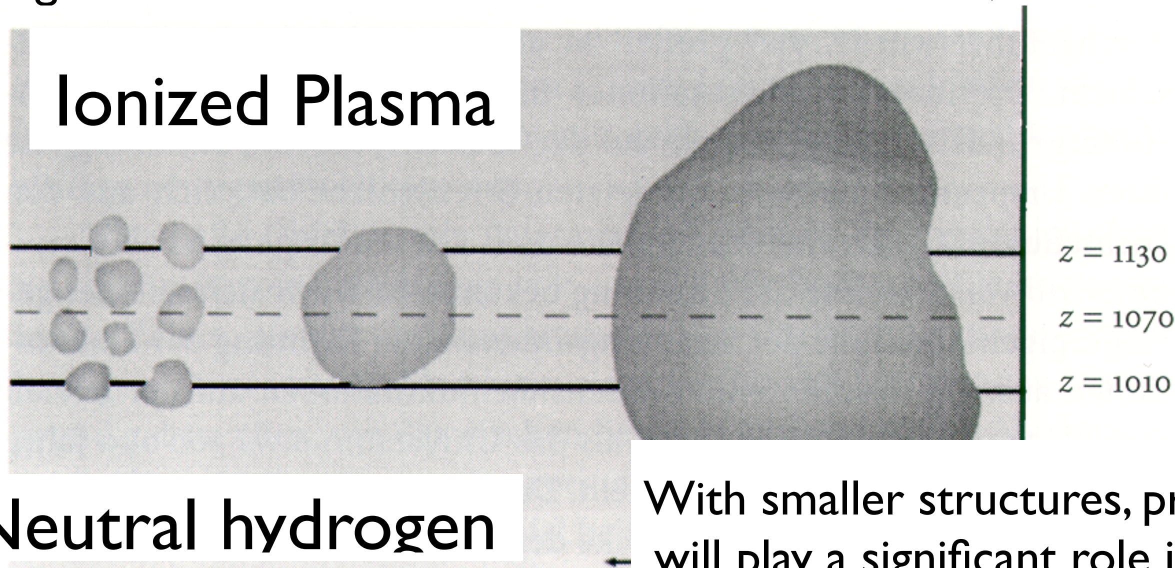
Ratio of Amplitude of 3rd to 1st Acoustic Peak: Matter Content

High Frequency Modes: Silk Damping...

What about the damping tail?

-- Decoupling does not happen instantaneously. This is not so important in viewing the last scattering surface for larger fluctuations. But for smaller fluctuations, the stuctures will overlap.

Note finite width of last scattering surface z = 1130 to 1010



Redshift

With smaller structures, projection effects will play a significant role in diluting signal

Silk Damping

Even before recombination, matter and radiation are not perfectly coupled: radiation leaks out of the perturbation, which leads to a dissipation of the perturbation.

This process occurs because photons bounce around (following a random walk) during recombination; for small scale fluctuations, the hot and cold photons can mix
=> on the scales corresponding to the distances photons can travel, the fluctuations are damped.

The dissipation scale $\lambda_D \sim 2c(\tau_{\gamma e}\, t)^{1/2}$ at time t

mean time before Thomson scattering: $\tau_{\gamma e} \propto n_e^{-1} \propto (1+z)^{-3}$

Before t_{eq}, $\lambda_D \propto (1+z)^{-5/2}$ since $t \propto (1+z)^{-2}$

After t_{eq}, $\lambda_D \propto (1+z)^{-9/4}$ since $t \propto (1+z)^{-1.5}$

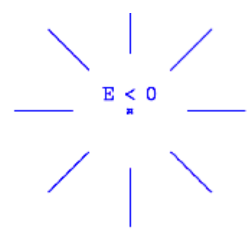
How CMB light can be broken down?

Measure Temperature and Polarization of Light

One tends to break down the polarization map into two modes
(Helmholtz-Hodge theorem)

90% of the photons in the CMB are unpolarized; this leaves 10% which is polarized.

E-modes



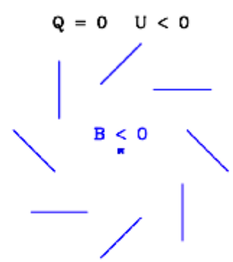
hot spots

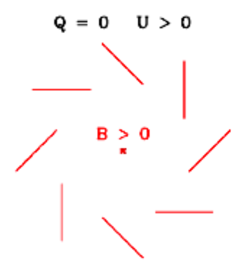
cold spots

E-modes are curl free and can be written as the gradient of a potential

$$\nabla \times E = 0$$

B-modes





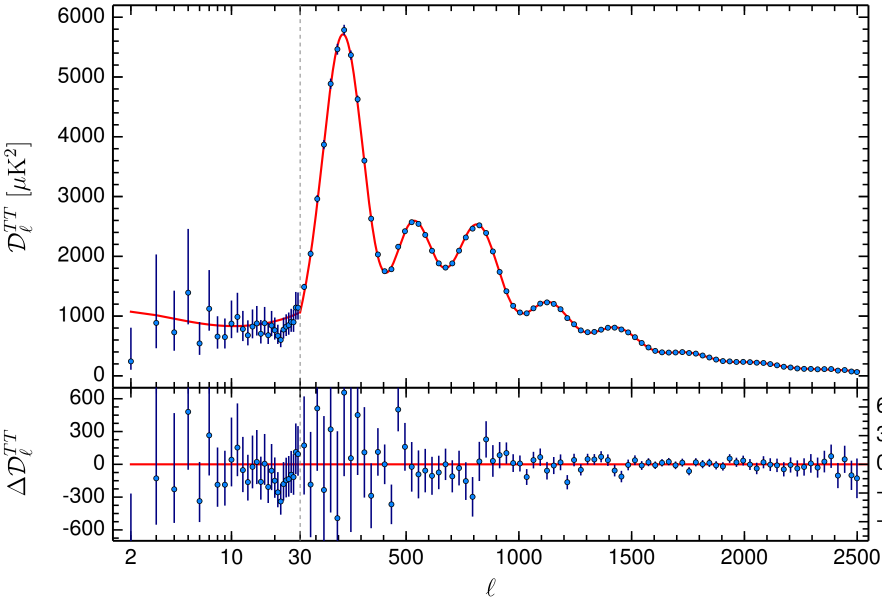
B-modes have no divergence.

$$\nabla \cdot B = 0$$

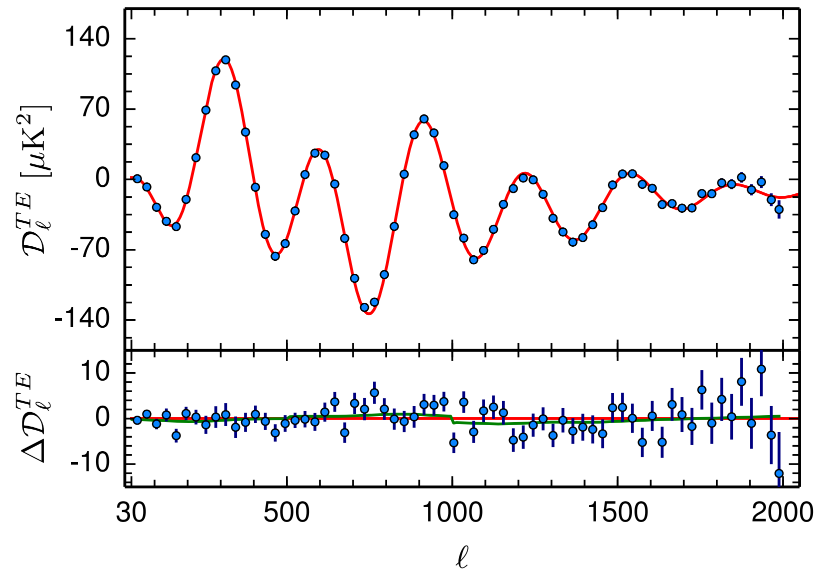
The terms E and B modes simply reflect the general form of the polarization fields and are in analogy with similar fields in electromagnetism. However, they have no direct relation with electric or magnetic fields

Cosmic Microwave Background

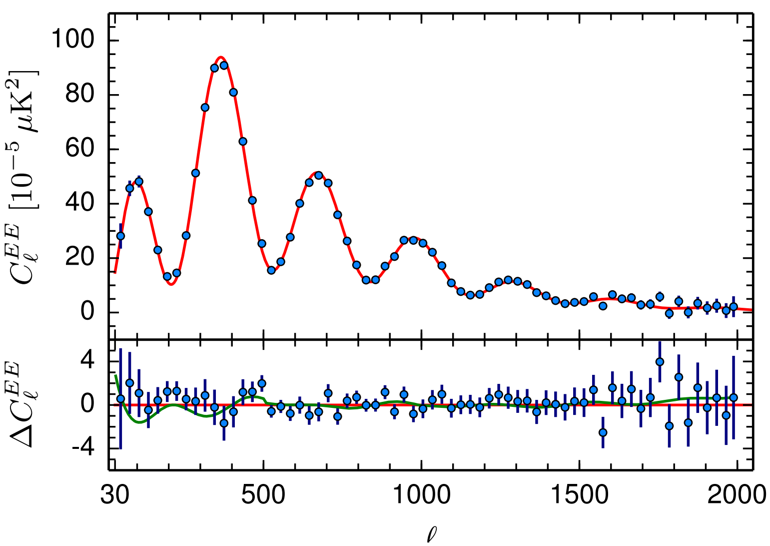
TT spectrum



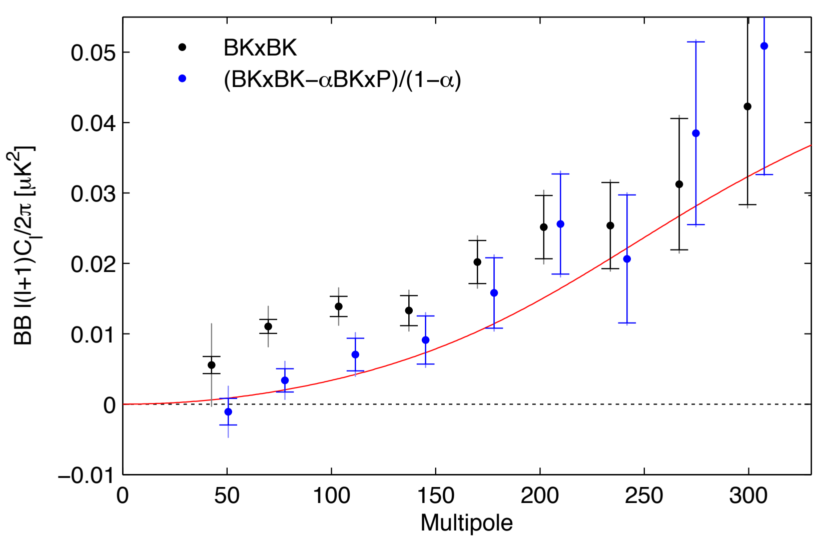
TE spectrum



EE spectrum

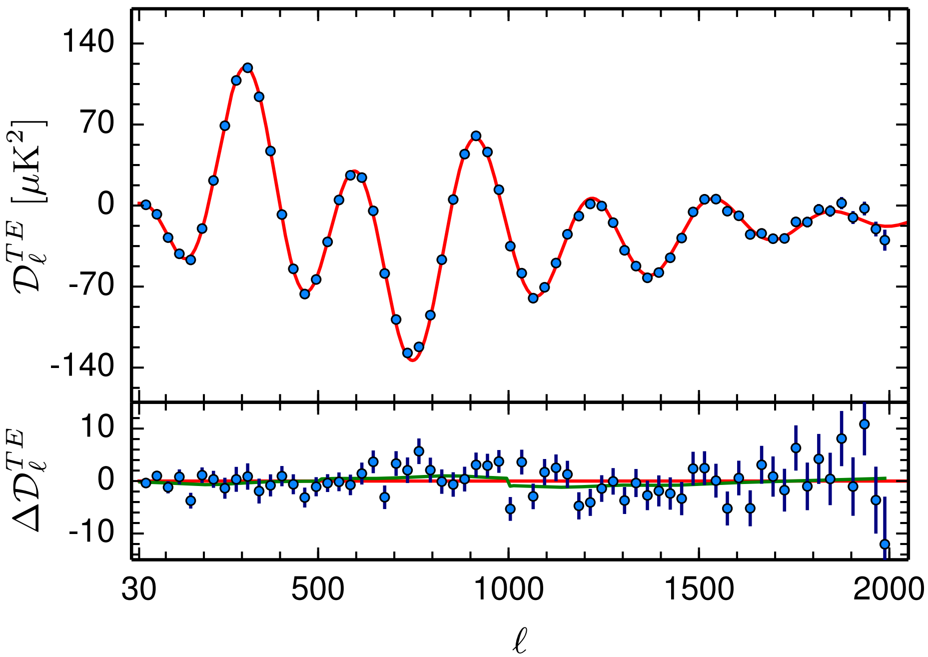


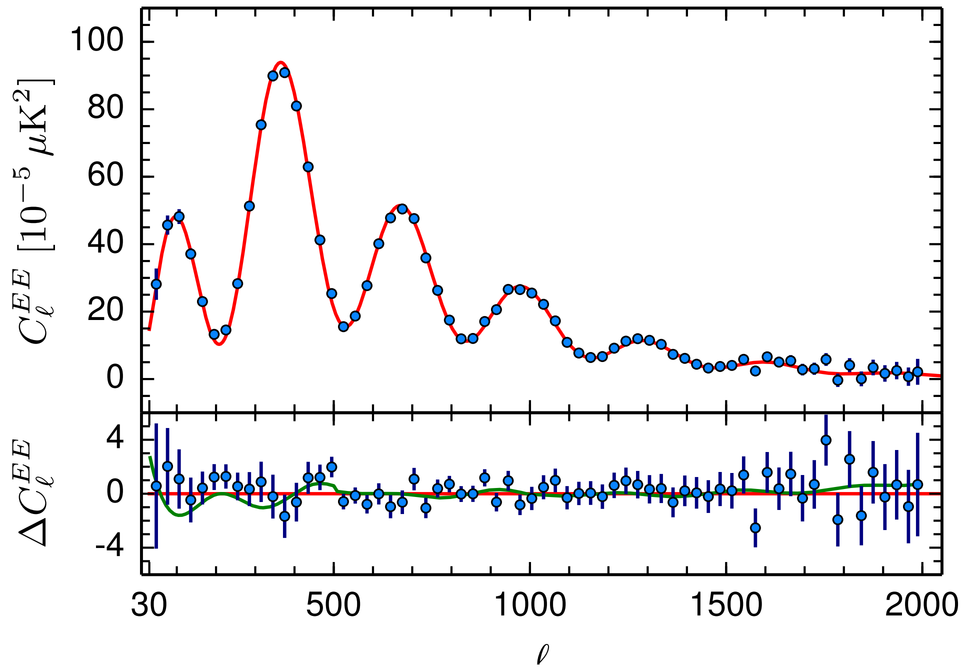
BB spectrum



Cosmic Microwave Background

TE / EE spectra



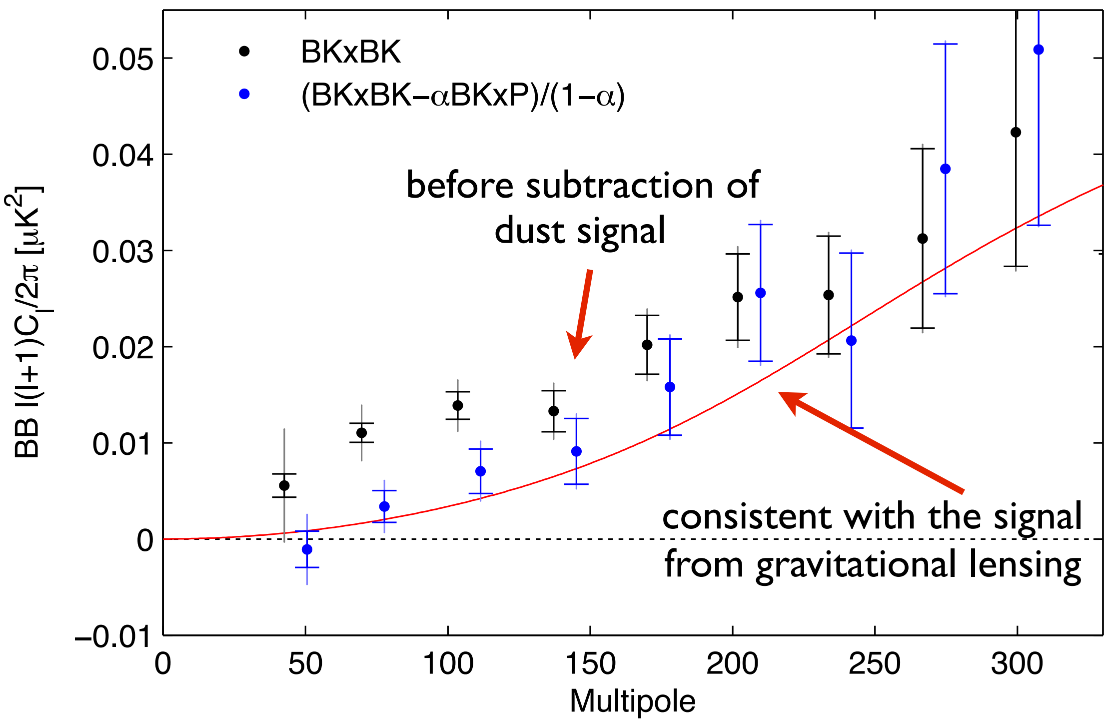


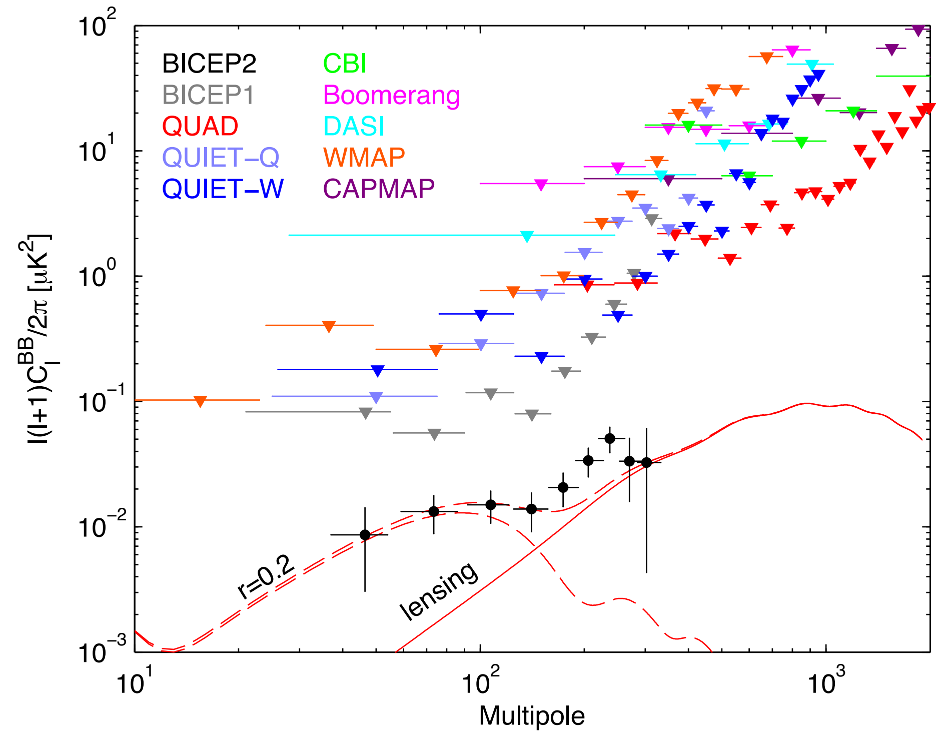
Contains Very Similar Information to that Present in TT Power Spectrum...

Allows us to verify that we understand the physics correctly...

Cosmic Microwave Background

BB spectra

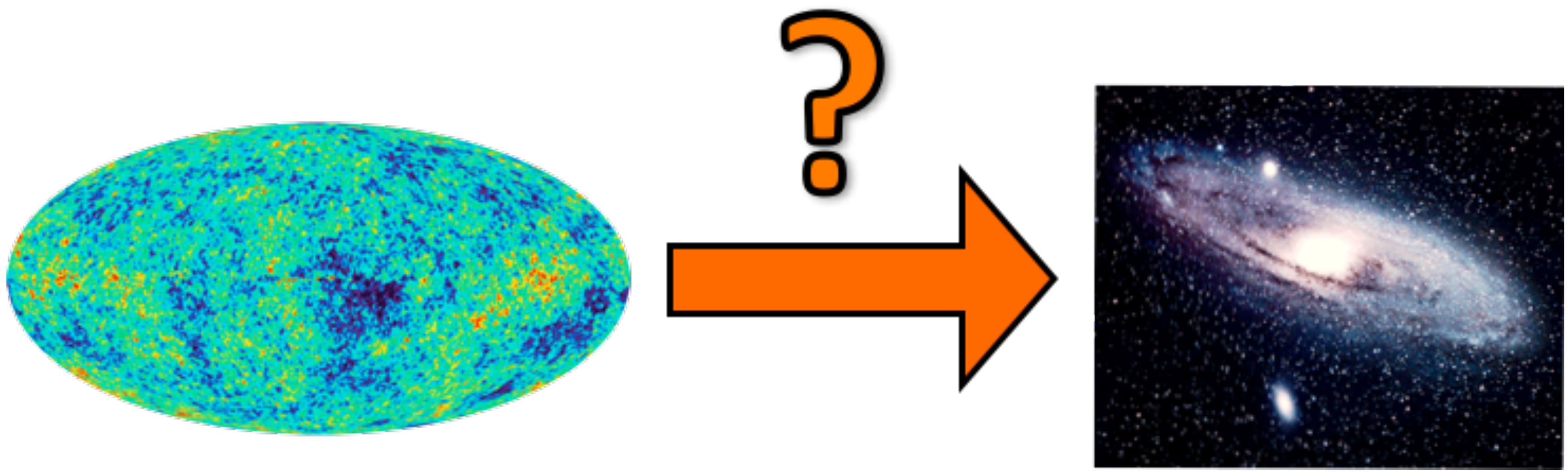




Signal arises from (1) gravity waves from inflation and (2) the impact of gravitational lensing on CMB...

Detection first reported in 2014 by BICEP II, but most of the signal likely from dust emission in our own galaxy

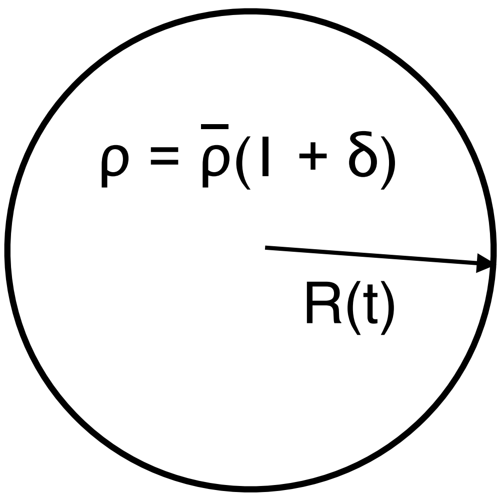
Gravitational instability



Growth of Structure

We start with tiny fluctuations in the background radiation temperature, which are related to density fluctuations. These grow into the very clustered universe we see today. We therefore need to study the density perturbations.

Consider a static, homogeneous matter-only Universe in which there is a spherical region that is overdense:



$$\delta(t) = (\rho - \bar{\rho})/\bar{\rho} << 1$$

$$d^2R/dt^2 = -G(\Delta M)/R^2 = -G((4/3)\pi R^3\bar{\rho}\delta)/R^2$$

$$d^2R/dt^2 / R = -(4\pi/3)G\bar{\rho}\delta$$

Hence, a mass excess $\delta > 0$ will cause the sphere to collapse

Conservation of mass gives $M = (4\pi/3)\bar{\rho}[1+\delta(t)]R(t)^3$ = constant during collapse

$R(t) = R_0\,[1+\delta]^{-1/3}$ where $R_0 = (3M/4\pi\rho)^{1/3}$

If $\delta << 1$, then $R(t) = R_0[1-(1/3)\delta(t)]$ => $d^2R/dt^2 = -(1/3)R_0 d^2\delta/dt^2$

mass conservation yields ==> $d^2R/dt^2 = -(1/3)R_0 d^2\delta/dt^2$ $(\delta<<1)$

$d^2\delta/dt^2 = 4\pi G\bar{\rho}\delta$ which has solutions $\delta = A_1 e^{t/t_{dyn}} + A_2 e^{-t/t_{dyn}}$

Growth of Structure

However, as the sphere collapses, pressure will build up. When a sphere is compressed by its own gravity, a pressure gradient will build up to counter the effects of gravity (e.g., in a star)

If the pressure gradient balances gravity, we have hydrostatic equilibrium.

The pressure gradient steepening takes time: any change in pressure travels with the speed of sound c_s; therefore the time to build up a pressure gradient is a sphere of radius R is pressure $\sim R/c_s$

$$c_s = c\,(dP/d\rho)^{1/2} = w^{1/2}\,c$$

For hydrostatic equilibrium to develop the gradient must build up before collapse:

$$R/c_s = t_{pressure} < t_{dyn} = (G\,\rho)^{-1/2} \Rightarrow R < c_s t_{dyn} = c_s/(G\bar{\rho})^{1/2} = \lambda_J$$

Jeans Length

A more accurate derivation yields $\lambda_J = c_s(\pi/G\rho)^{1/2} = 2\pi c_s\, t_{dyn}$

Consider a spatially flat Universe with mean density ρ => $1/H = (3/8\pi G\bar{\rho})^{1/2} = (3/2)^{1/2}\, t_{dyn}$

The Jeans length in an expanding universe will then be

$$\lambda_J = 2\pi c_s t_{dyn} = 2\pi(2/3)^{1/2} c_s/H$$

$= 1.22\ t_{dyn}$

If we now focus on one component with equation of state w and $c_s = (w)^{1/2}\, c$

$$\lambda_J = 2\pi c_s t_{dyn} = 2\pi(2/3)^{1/2}\, w^{1/2}\, c/H$$

For a photon gas, $c_s = c/3^{1/2} \sim 0.58c$ => $\lambda_J = 2\pi c_s t_{dyn} = 2\pi\, 2^{1/2}\, c/3H \sim 3c/H$

Growth of Structure

Density fluctuations in the radiative component will be pressure supported if they are smaller than 3 times the Hubble radius

Such fluctuations will oscillate. Only larger ones will collapse.

To get collapsed structures, we need a non-relativistic component with $w^{1/2} << 1$

Prior to decoupling, the baryons were coupled to the photons => no collapse possible

At $c/H(z_{rec}) \sim 0.2$ Mpc and $\varepsilon_\gamma = 1.4\ \varepsilon_{baryon}$

=> λ_J (before decoupling) = $3c/H(z_{rec}) = 0.6$ Mpc

baryon Jeans mass before decoupling = $7 \times 10^{18}\ M_{sun}$

30,000x coma cluster

After decoupling, we have two separate gases: for baryons, the sound speed drops to

c_s (baryon) = $(kT/mc^2)^{1/2}\ c$

c_s (baryon) = $(1.5 \times 10^{-5})c$ or 5 km/s

$kT = 0.26$ eV

$mc^2 = 1.22\ m_p\ c^2 = 1140$ MeV

$Y = 0.24$

Then, after decoupling, the Jeans length decreased by a factor of c_s(baryon)/c_s(photon) ~ 2.6×10^{-5}

M_J (after) ~ $10^5\ M_{solar}$

much smaller than the mass of our galaxy, ~baryonic mass of the smallest dwarf galaxies

After decoupling, baryon density perturbations could start growing.

Growth of Structure

We can study the Jeans theory in a bit more detail, focusing first on the collisional fluids. The equations of motion are in the Newtonian approximation:

$$\partial\rho/\partial t + \nabla \cdot (\rho\mathbf{v}) = 0 \qquad \text{continuity equation}$$

$$\partial\mathbf{v}/\partial t + (\mathbf{v} \cdot \nabla)\mathbf{v} + (1/\rho)P + \nabla\Phi = 0 \qquad \text{Euler equation}$$

$$\nabla^2\Phi - 4\pi G\rho = 0 \qquad \text{Poisson's Equation}$$

We will also neglect any dissipative terms arriving from viscosity or Thermal conductivity. Therefore, we have conservation of entropy per unit mass:

$$\partial S/\partial t + \mathbf{v} \cdot \nabla S = 0$$

A trivial solution is the following: $\rho = \rho_0, \mathbf{v} = 0, S = S_0, p = p_0, \nabla\Phi = 0$

Note that if $\rho = \rho_0 \neq 0$, then Φ must vary spatially => homogeneous distribution of ρ cannot be stationary, similar to what we saw when we derived the Friedmann equation

Although the derivation is formally incorrect, the results are qualitatively unchanged and the results can be "reinterpreted" to give correct results.

$$\rho = \rho_0 + \delta\rho, \qquad \mathbf{v} = \delta\mathbf{v}, \qquad p = p_0 + \delta p \qquad S = S_0 + \delta S \qquad \Phi = \Phi_0 + \delta\Phi$$

$$\partial\delta\rho/\partial t + \rho_0\nabla \cdot (\delta\mathbf{v}) = 0$$

$$\partial\delta\mathbf{v}/\partial t + (1/\rho_0)(dp/d\rho)_s\nabla\delta\rho + (1/\rho_0)(dp/ds)_\rho\nabla\delta s + \nabla\delta\phi = 0$$

$$\nabla^2\delta\phi - 4\pi G\delta\rho = 0 \qquad \partial\delta S/\partial t = 0$$

Modeling the Growth of Structure using Waves in Fluid

We will look for solution in the form of plane waves $\delta u_i = \delta_i\, e^{i\mathbf{k}\cdot\mathbf{r}}$ where $\delta u_i = \delta\rho, \delta v, \delta\phi, \delta s$

Given that the unperturbed solution do not depend on position, we can search for solutions:

$\delta_i(t) = \delta_{0,i}\, e^{i\omega t}$ amplitude D, V, Φ, Σ

Use that $c_s = (\partial P/\partial\rho)_s$ and $\delta_0 = D/\rho_0$

$$\Rightarrow \omega\,\delta_0 + \mathbf{k}\cdot\mathbf{V} = 0$$

$$\Rightarrow \omega\,\mathbf{V} + \mathbf{k}\,c_s^2\,\delta_0 + \mathbf{k}/\rho\,(dp/ds)_\rho\,\Sigma + \mathbf{k}\Phi = 0$$

$$\Rightarrow k^2\,\Phi + 4\pi G\rho_0\delta_0 = 0$$

$$\Rightarrow \omega\,\Sigma = 0$$

Let us consider solutions with $\omega \neq 0 \Rightarrow \Sigma = 0$: perturbations are adiabatic

Also, $\mathbf{k}\cdot\mathbf{V} \neq 0$, we can decompose into components parallel and perpendicular to $\mathbf{V}$

$\mathbf{k}$ perpendicular to $\mathbf{V} \Rightarrow \delta_0 = 0, \Phi = 0$, these vertical models do not imply a density perturbation.

$\mathbf{k}$ is parallel to $\mathbf{V}$

$$\Rightarrow \omega\,\delta_0 + kV = 0$$

$$\Rightarrow \omega V + kc_s^2\,\delta_0 + k\Phi = 0$$

$$\Rightarrow k^2\,\Phi + 4\pi G\rho_0\delta_0 = 0$$

$$\begin{pmatrix} \omega & k & 0 \\ kc_s^2 & \omega & k \\ 4\pi G\rho_0 & 0 & k^2 \end{pmatrix}\begin{pmatrix} \delta_0 \\ V \\ \Phi \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

This admits a non-zero solution for δ_0, V, Φ if and only if its determinant vanishes $\Rightarrow \omega$ and k must satisfy the dispersion relation

$$\Rightarrow \omega^2 - c_s^2\,k^2 + 4\pi G\rho_0 = 0$$

New Material for This Week

Modeling the Growth of Structure using Waves in Fluid

We will look for solution in the form of plane waves $\delta u_i = \delta_i\, e^{i\mathbf{k}\cdot\mathbf{r}}$ where $\delta u_i = \delta\rho, \delta v, \delta\phi, \delta s$

Given that the unperturbed solution do not depend on position, we can search for solutions:

$\delta_i(t) = \delta_{0,i}\, e^{i\omega t}$ amplitude D, V, Φ, Σ

The solutions are of two types, depending on whether $\lambda = 2\pi/k$ larger or smaller than

$$\lambda_J = c_s\, (\pi/G\rho_0)^{1/2}$$

In the case that $\lambda < \lambda_J$, the value of ω is real and $\omega = \pm c_s\, k[1-(\lambda/\lambda_J)^2]^{1/2}$

These represent two sound waves in directions $\pm\mathbf{k}$ with a dispersion ω

If $\lambda > \lambda_J$, the frequency is imaginary: $\omega = \pm\, i\, (4\pi G\rho_0)^{1/2}\, [1-(\lambda_J/\lambda)^2]^{1/2}$

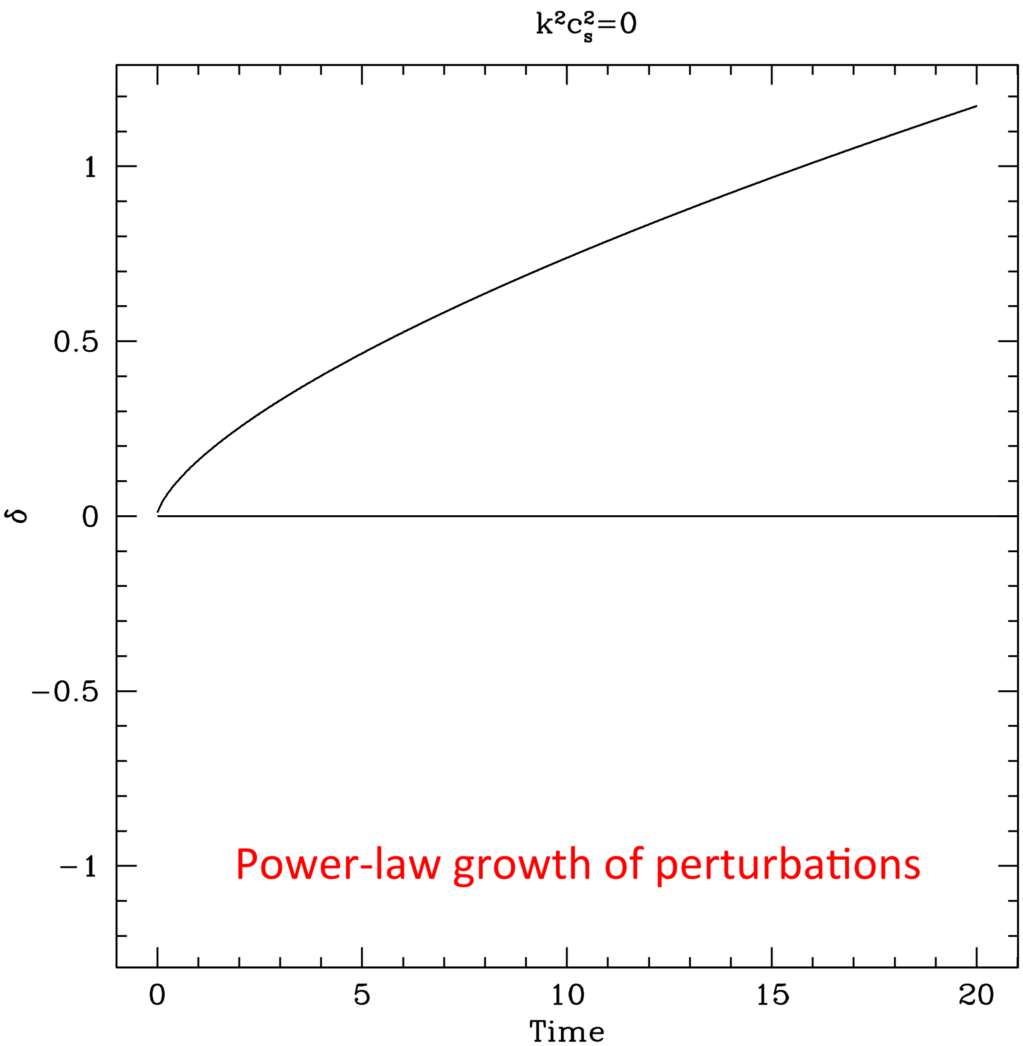
and the solution for the density is $d\rho/\rho_0 = \delta_0\, e^{i\mathbf{kr}}\, e^{\pm\omega t}$

The characteristic time scale for the evolution of the amplitude is

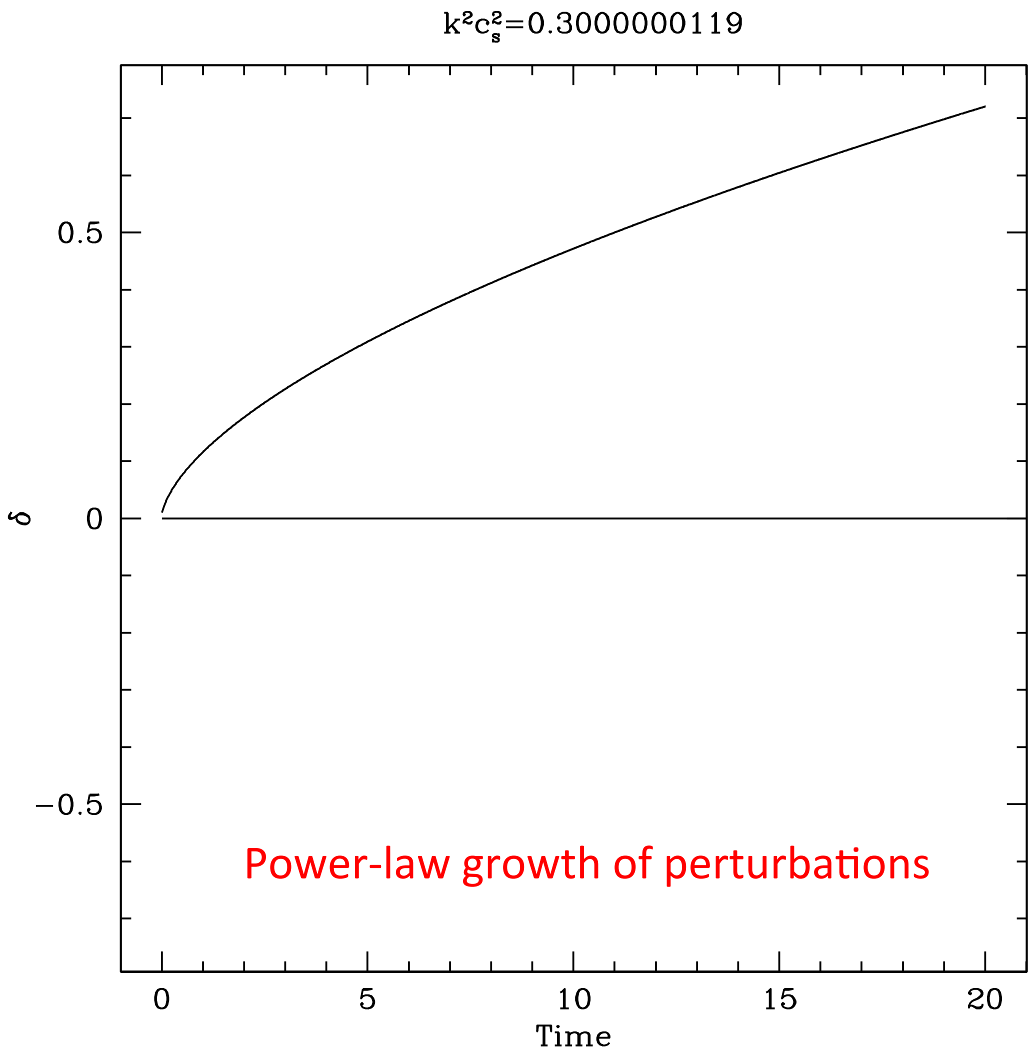
$$\tau = \omega^{-1} = 1/(4\pi G\rho_0)^{1/2}\, [1-(\lambda_J/\lambda)^2]^{-1/2}$$

for $\lambda >> \lambda_J$, this corresponds to the dynamical or free-fall time.

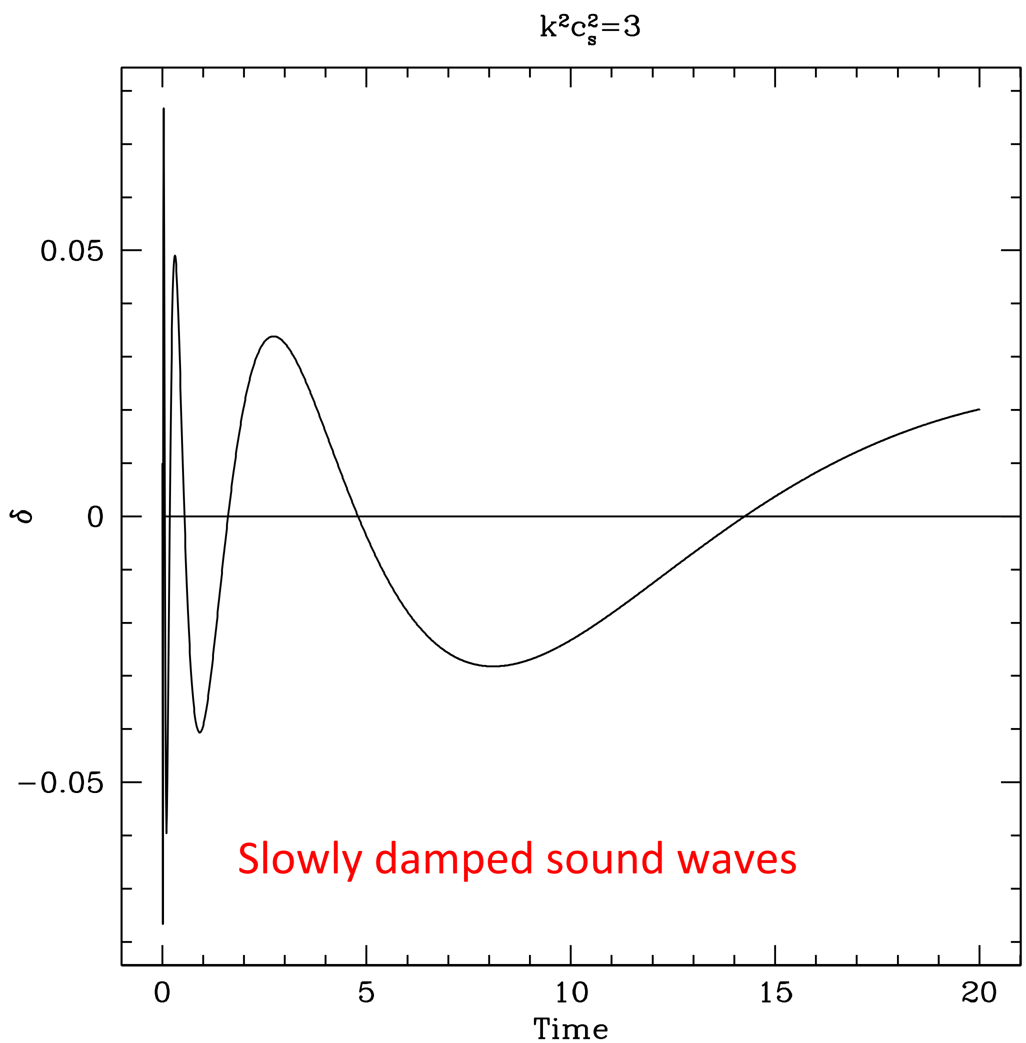
Gravitational instability



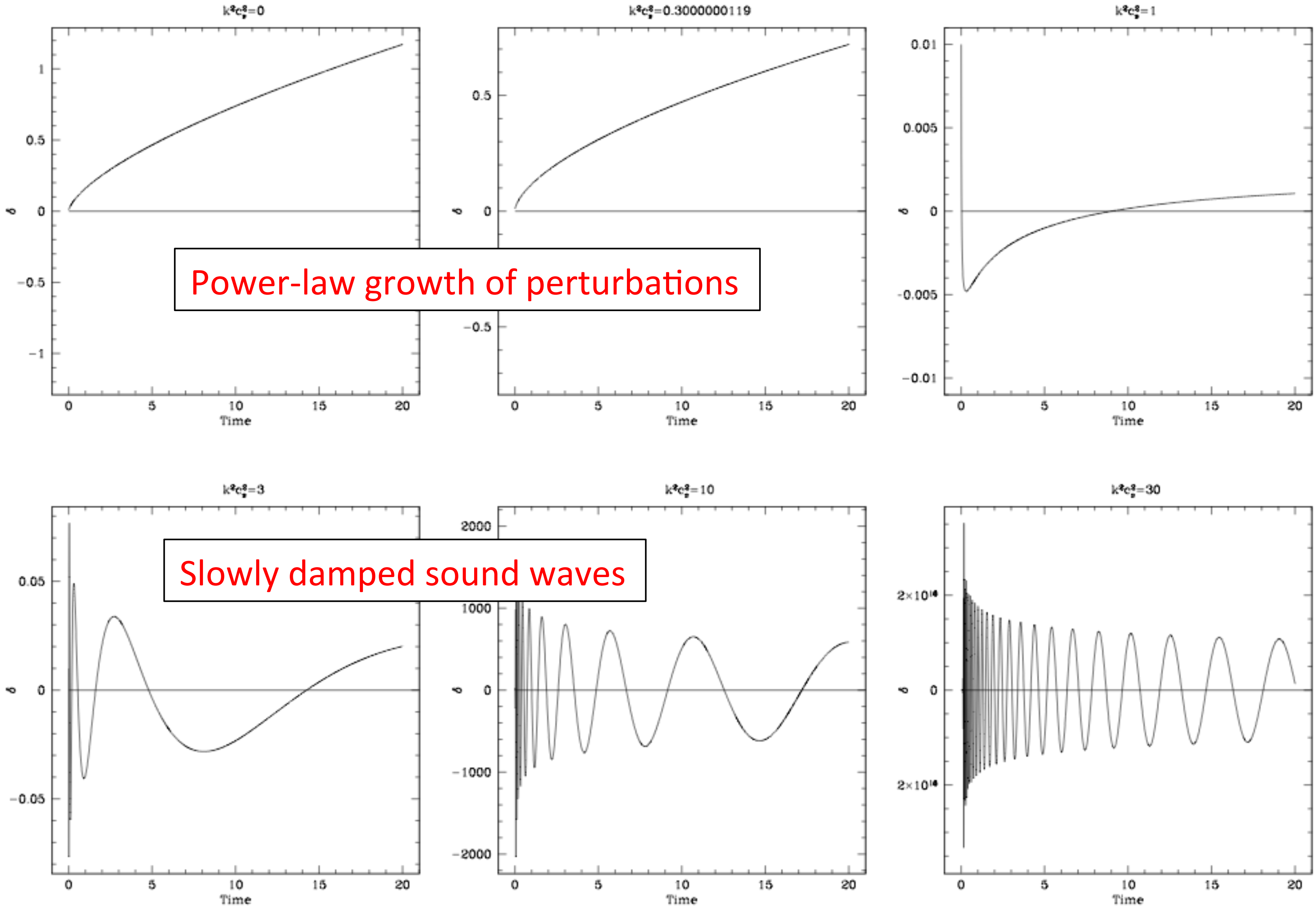
Gravitational instability



Growth of structure



Growth of structure



Growth of Structure in Expanding Universe

Let us now look at the homogeneously expanding solution with expansion faction a(t)

$$\rho_{bg} = \rho_0\ (a/a_0)^{-3},\ \ \mathbf{v}_{bg} = ((da/dt)/a)\mathbf{r}, \Phi_{bg} = (2/3)\pi G\rho_{bg} r^2, p_{bg} = p(\rho_{bg})$$

We again perturb the background using

$$d\rho_{bg}/dt = -3((da/dt)/a)\rho_{bg};\ \nabla \cdot \mathbf{v}_{bg} = 3((da/dt)/a); (dv \cdot \nabla)\mathbf{v}_{bg} = ((da/dt)/a)d\mathbf{v}$$

$$\Rightarrow \partial\delta\rho/\partial t + \rho_0 \nabla \cdot (\delta\mathbf{v}) = 0$$

$$\partial\delta\mathbf{v}/\partial t + (1/\rho_0)(dp/d\rho)_s \nabla\delta\rho + (1/\rho_0)(dp/ds)_\rho \nabla\delta s + \nabla\delta\Phi = 0$$

$$\nabla^2\delta\Phi - 4\pi G\delta\rho = 0$$

We can drop the $\mathbf{r} \cdot \nabla$ terms since they are a coordinate dependent artifact of the Newtonian formulation

As before, $\delta u_i = u_i(t)e^{i\mathbf{kr}}$ (note $k = (2\pi/\lambda) = (2\pi/\lambda_0)(a_0/a) = k_0\ (a_0/a)$)

$$\Rightarrow dD/dt + 3((da/dt)/a)\ D + i\rho_{bg}\ (\mathbf{k} \cdot \mathbf{V}) = 0$$

$$d\mathbf{v}/dt + ((da/dt)/a)\mathbf{v} + ic_s^2\ \mathbf{k}(D/\rho) + i\ \mathbf{k} \cdot \Phi = 0$$

$$k^2\,\Phi + 4\pi G\ D = 0$$

$\rho_{bg}\ \delta$

$$\Rightarrow dD/dt + 3((da/dt)/a)\ D + i\rho_{bg}\ kV = 0$$

$$dV/dt + ((da/dt)/a)V + ik(c_s^2 - 4\pi G\rho_{bg}/k^2)D/\rho = 0$$

as $D = \rho_{bg}\,\delta$ $\quad (dD/dt = d\rho_{bg}/dt\ \delta + \rho_{bg}\, d\delta/dt)$ $\quad d\delta/dt + ikV = 0$

which upon differentiation gives $\quad d^2\delta/dt^2 + ik(dV/dt - (da/dt)V/a) = 0$

$$\Rightarrow d^2\delta/dt^2 + 2((da/dt)/a)\ d\delta/dt + (c_s^2k^2 - 4\pi G\rho)\delta = 0$$

Growth of Structure in Expanding Universe

$$\Rightarrow d^2\delta/dt^2 + 2((da/dt)/a)\, d\delta/dt + (c_s^2k^2 - 4\pi G\rho)\delta = 0$$

This is a generalization of the static case and gives the evolution of perturbations of waves with wavenumber k as long as $\delta \ll 1$

To solve this equation, we need a prescription of a, ρ, and c_s

* Flat matter-dominated Einstein-de Sitter model

$\rho = 1/(6\pi G t^2)$ $\quad\quad$ $a = a_0\,(t/t_0)^{2/3}$ $\quad\quad$ $(da/dt)/a = H = (2/3t)$

$$\Rightarrow d^2\delta/dt^2 + (4/3)(d\delta/dt)/t - (2/3t^2)[1 - c_s^2k^2/4\pi G\rho]\delta = 0$$

If we assume that matter compromises monoatomic particles of mass m, then the sound speed is

$$c_s = (5k_BT_m/3m)^{1/2} = (5k_BT_{0,m}/3m)^{1/2}\,(a_0/a)$$

In this case that c_sk is very small (long wavelengths, low sound speed)

$$\Rightarrow d^2\delta/dt^2 + (4/3)(d\delta/dt)/t - (2/3t^2)\delta = 0$$

Try a solution $\delta \propto t^n$

$$\Rightarrow [n(n-1) + (4/3)n - 2/3]t^{n-2} = 0$$

$$\Rightarrow n(n-1) + (4/3)n - 2/3 = 0 \quad\quad \Rightarrow n=-1 \text{ or } n=2/3$$

growing mode $\delta_+ \propto t^{2/3} \propto a$; decaying mode $\delta_- \propto t^{-1}$

The densities grow $\delta \propto t^{2/3} \propto a(t) \propto 1/(1+z)$ as long as $\delta \ll 1$

Growth of Structure in Expanding Universe: Large k

For large k (short wavelength) and under the assumption that c_s varies slowly

$$d^2\delta/dt^2 + (4/3)(d\delta/dt)/t - (2/3\rho)(1 - c_s^2k^2/4\pi G)\delta = 0$$

looks like a damped harmonic oscillator with frequency 1/t

If we try again $\delta \propto t^n$ we find solutions $n^2 + (n/3) - (2/3)[1 - c_s^2k^2/4\pi G\rho] = 0$

$$n = -(1/6) \pm (1/6)(25 - 6c_s^2k^2/\pi G\rho)^{1/2} = 0$$

hence instability when $k <\sim (G\rho)^{1/2}/c_s^2$ and oscillations for larger k

Once more the Jeans criterion

Growth of Structure: Open/Lambda/Radiation Universes

Open

- If we consider a low Ω_m where curvature dominates, then $a \propto t$ and

and $d^2\delta/dt^2 + 2d\delta/dt/t = 0$, which has solutions $\delta \propto t^{-1}$ and $\delta \propto t^0$

i.e., no growth in a low density universe

Lambda

- If we consider a lambda-dominated universe, then

and $d^2\delta/dt^2 + 2(da/dt/a)\, d\delta/dt = 0$, with solutions $\delta_- \propto e^{-2Ht}$ and $\delta_+ \propto t^0$

i.e., no growth in a lambda-dominated universe

Radiation

- In the case of a radiation dominated universe, the derivation needs to include the pressure in the energy density $\rho \rightarrow \rho + P/c^2$ and one can show that

$$d^2\delta/dt^2 + 2((da/dt)/a)d\delta/dt + [c_s k^2 - 32/3\pi\ G\rho]\delta = 0$$

For a radiation dominated universe where $a \propto t^{1/2}$ and $\rho = 3/32\pi G t^2$

$$\Rightarrow d^2\delta/dt^2 + (d\delta/dt)/t - (1/t^2)[1 - 3c_s^2k^2/32\pi G\rho]\delta = 0$$

For $k \rightarrow 0$, the solution $\delta \propto t^n$ with $\delta_+ \propto t^1$ and $\delta_- \propto t^{-1}$

As before, damped oscillations for large k, with a transition near the Jeans Length

How does structure grow?

	above horizon	below horizon	z
radiation dominated epoch	$\delta \propto a^2$ $P \propto a^4$	no growth	
			3500
matter dominated epoch	$\delta \propto a$ $P \propto a^2$	$\delta_{DM} \propto a$, $P_{DM} \propto a^2$, oscillations in baryons	
			1100
after decoupling (if Ω_m~1)	$\delta \propto a$ $P \propto a^2$	$\delta \propto a$ $P \propto a^2$	

Growth of Structure at Early Times (before matter-radiation equality)

Consider now the growth of matter perturbations in a Universe where expansion is driven by a relativistic component.

Assume $k = 0$ => $d^2\delta/dt^2 + 2((da/dt)/a)d\delta/dt - 4\pi G\rho_m\delta = 0$

We already examined the evolution for $t >> t_{eq}$ (matter-radiation equality)

But at earlier times a and ρ evolve differently!

If we define $y = \rho_m/\rho_r = a/a_{eq}$ increases with time; $y = 1$ at $z = z_{eq} \sim 3500$

$\delta = \delta\rho_m/\rho_m$; rewrite the perturbation equation from a function of time in one of y

$d\delta/dt = \delta' (da/dt)/a_{eq}$ $\quad d^2\delta/dt^2 = [\delta''(da/dt)^2 + \delta'(d^2a/dt^2)]/a_{eq}^2$ $\quad \delta' = (d/dy)\delta$

$\rho_m = y/(1+y)\ \rho$ and $\rho_r = y/(1+y)\ \rho$ and $p = (1/3)\rho_r c^2$

Friedmann's Equation: $((da/dt)/a)^2 = (8\pi G/3)(\rho_m + \rho_r)$

Acceleration Equation: $d^2a/dt^2 = (-4\pi G/3)(\rho + 3P/c^2)a = (-4\pi G/3)(\rho + \rho/(1+y))a = -(1/2)(2+y)((da/dt)/a)^2/(1+y)$

Growth of Structure

Then $d^2\delta/dt^2 + 2((da/dt)/a)d\delta/dt - 4\pi G\rho_m\delta = 0$ can be rewritten as

$$\delta'' + (2+3y)\delta'/2y/(1+y) - 3\delta/2y/(1+y) = 0$$

Has 2 solutions: one growing and one decaying. The growing mode:

$$\delta_+ \propto 1+(3/2)y \sim 1 + 5000/(1+z)$$

Before z_{eq}, we have that $y < 1$ and the growing mode is frozen. This Meszaros effect applies to cold dark matter density fluctuations (not coupled to the radiation via pressure) on large scales.

The total growth from $t = 0$ to t_{eq} is $\delta_+(y=1)/\delta_+(y=0) = 5/2$ and afterwards by another factor $1+z_{eq}$

The physical reason for this slow growth is that before t_{eq} the Jeans time is longer than the expansion time. The energy in radiation causes the Universe to expand so fast that the matter has no time to respond.

Growth of Structure

Before decoupling, the dark matter grows normally, i.e., $\delta_{DM} \propto a$

but the baryon dynamics are coupled to that of the radiation.
=> δ_{bary} oscillates like the radiation,

but after t_{eq},

Let us consider the evolution of δ_{DM} and δ_{bar} distinctly:

$$d^2\delta_{bar}/dt^2 + (4/3t)d\delta_{bar}/dt = 4\pi G(\bar{\rho}_{bar}\,\delta_{bar} + \bar{\rho}_{DM}\,\delta_{DM})$$

$$d^2\delta_{DM}/dt^2 + (4/3t)d\delta_{DM}/dt = 4\pi G(\bar{\rho}_{bar}\,\delta_{bar} + \bar{\rho}_{DM}\,\delta_{DM})$$

If we use that $\delta_m = (\bar{\rho}_{bar}\,\delta_{bar} + \bar{\rho}_{DM}\,\delta_{DM})/(\bar{\rho}_{bar} + \bar{\rho}_{DM}) \sim \delta_{DM}$ and $\Delta \equiv (\delta_{DM} - \delta_{bar})$

$$d^2\Delta/dt^2 + (4/3t)d\Delta/dt = 0 \qquad\qquad \Rightarrow \Delta = \text{constant or } \Delta \propto t^{-1/3}$$

$\delta_m \propto t^{2/3} \propto a$

$$\delta_{DM} / \delta_{bar} = (\rho_m\,\delta_m + \rho_{bar}\,\Delta)/(\rho_m\,\delta_m - \rho_{DM}\,\Delta) \rightarrow 1$$

The initial non-zero value of δ_{bar} at decoupling leaves a small effect on δ_m at later times => these are the baryon acoustic oscillations.

Growth of structure

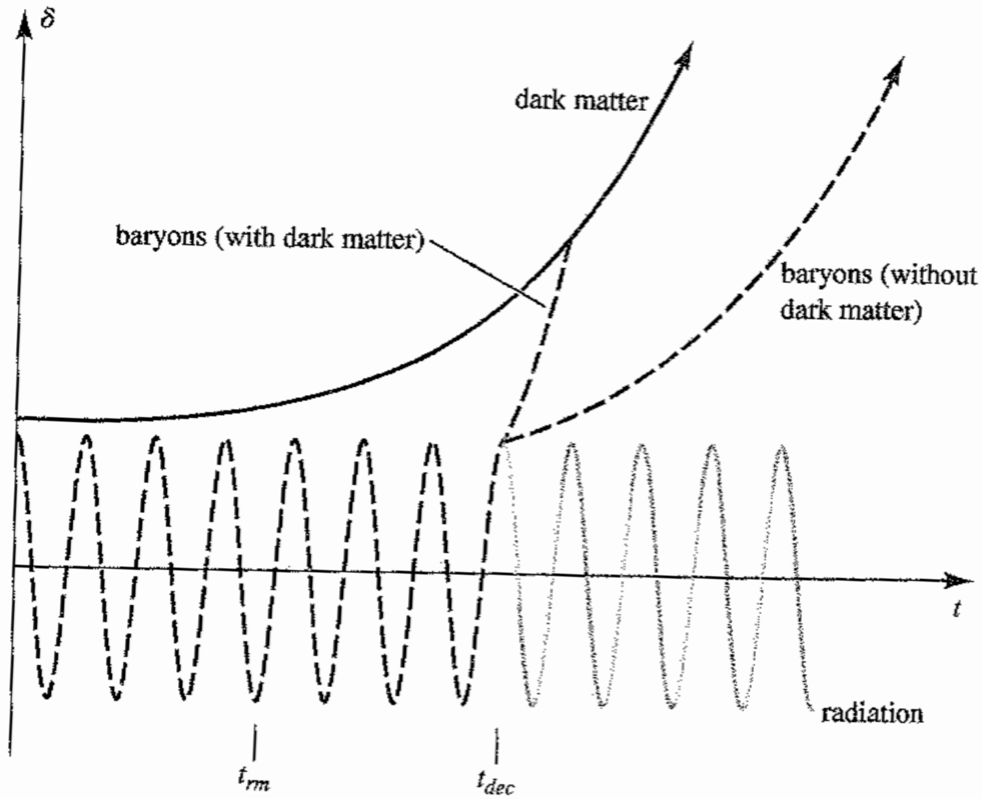
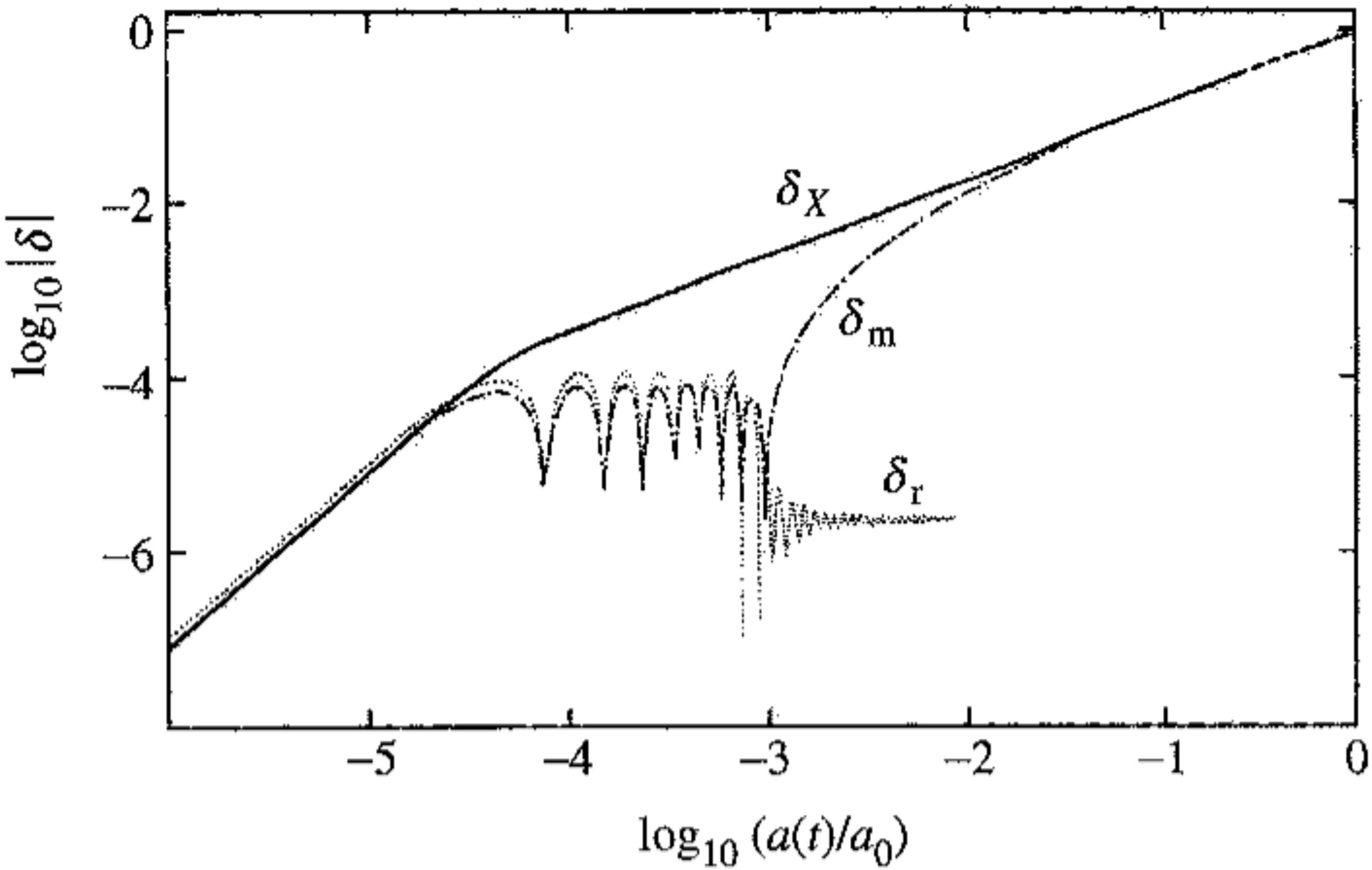


FIGURE 4 A highly schematic drawing of how density fluctuations in different components of the universe evolve with time.

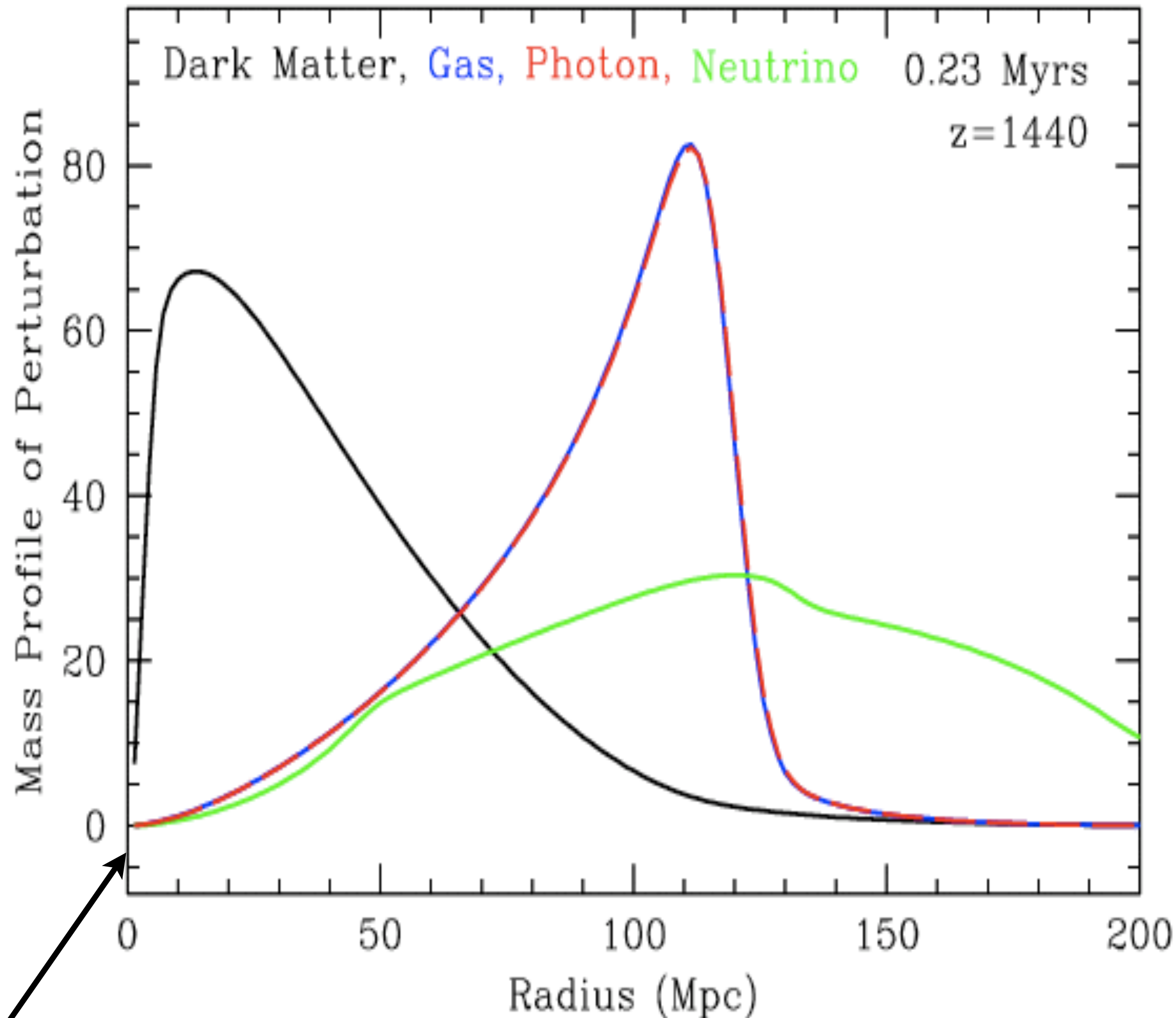
from "Introduction to Cosmology" (Ryden, 2014)

Growth function



From “Cosmology” Coles & Lucchin, 2nd edition

Baryons + DM affect each other after decoupling: How?

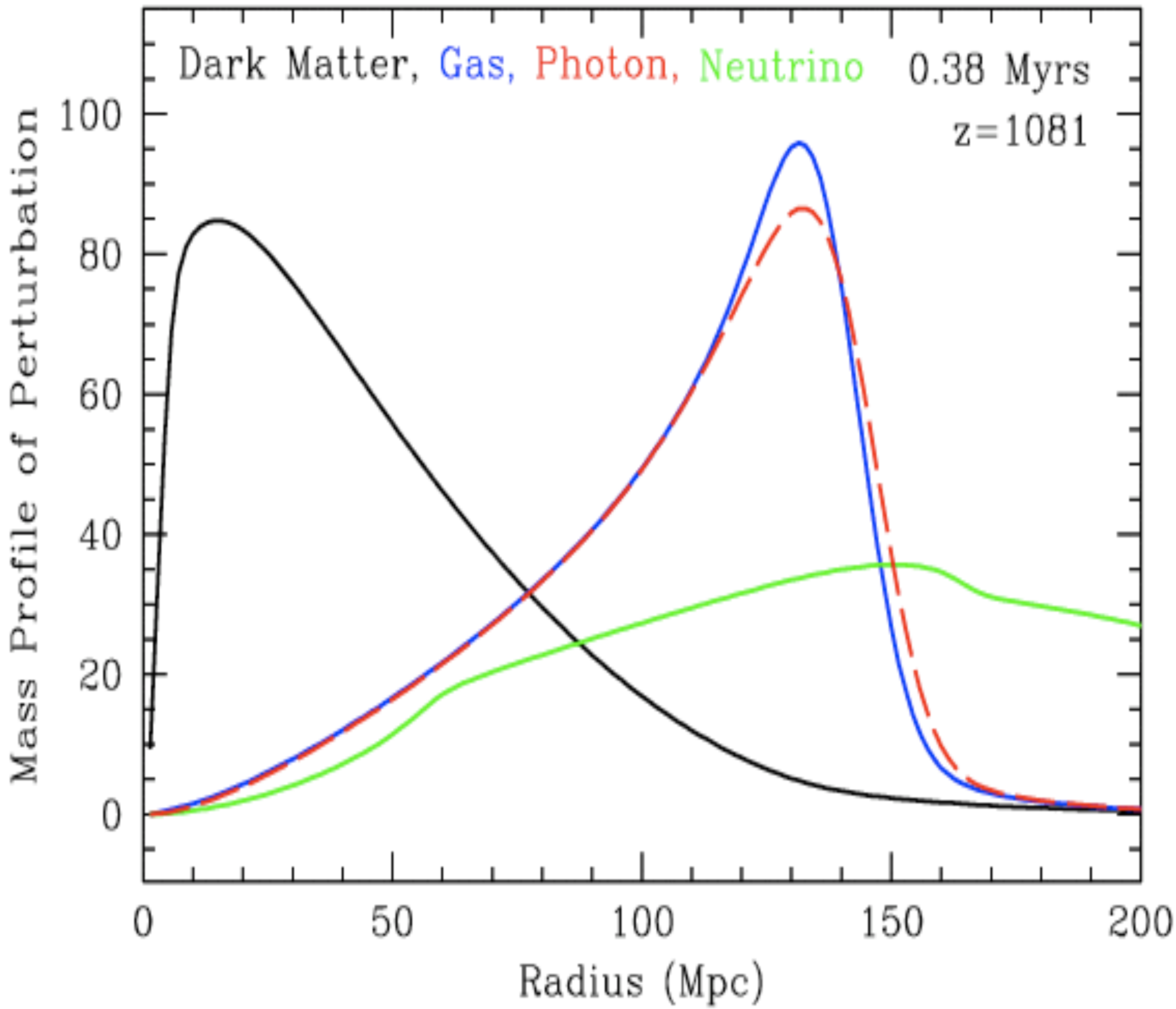


imagine we have a overdensity here at time t = 0

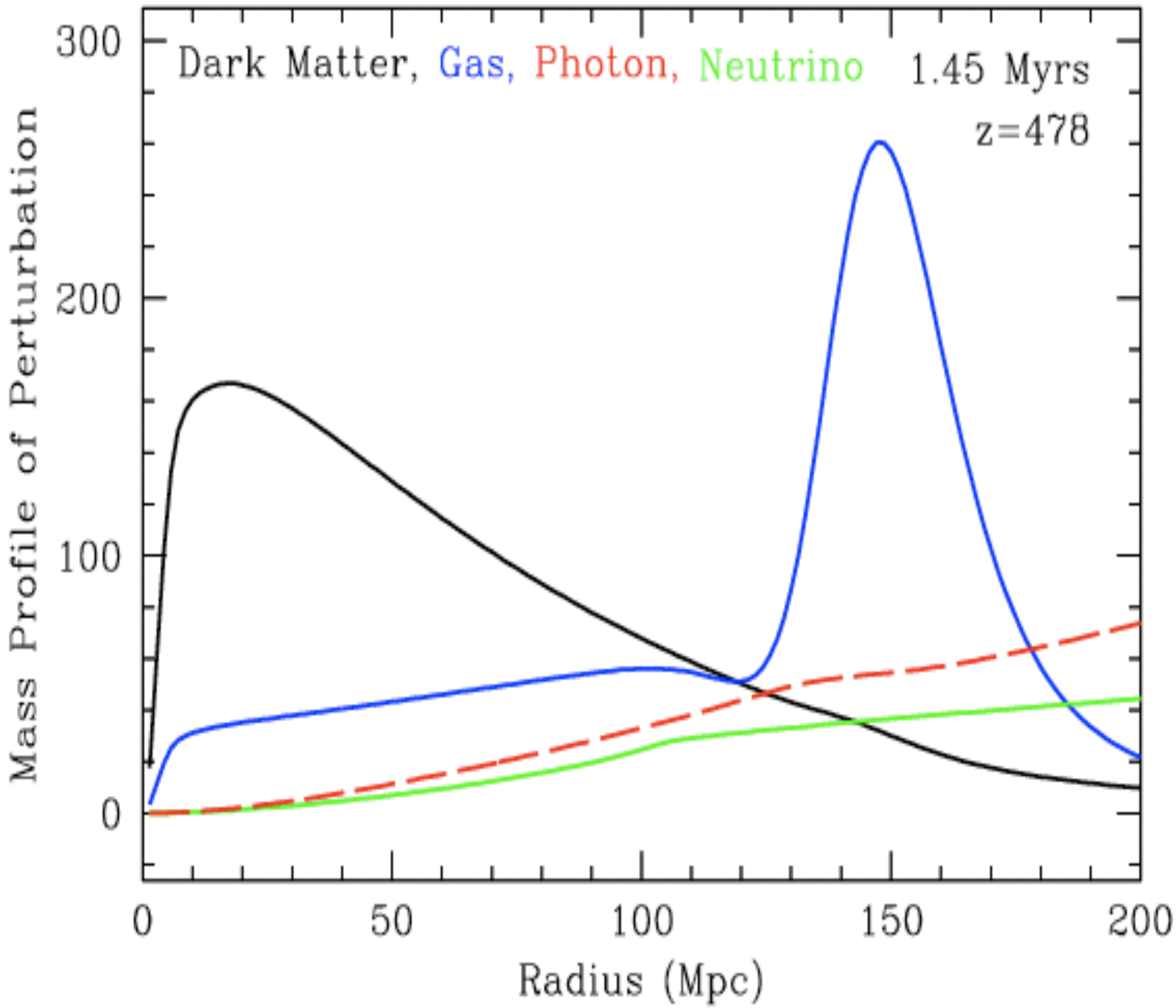
dark matter will fall towards it

but baryons and radiation will bounce

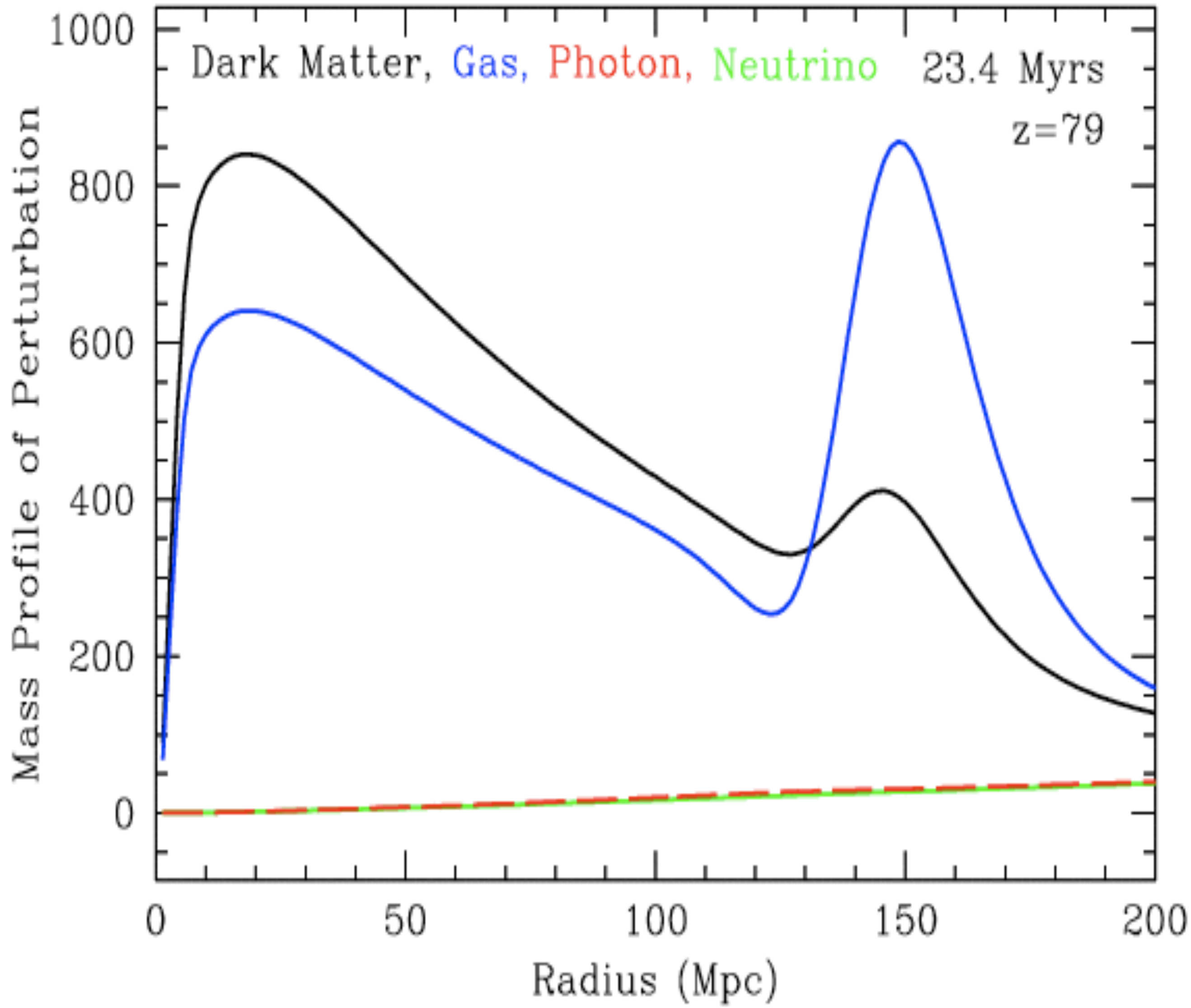
Baryons + DM affect each other after decoupling: How?



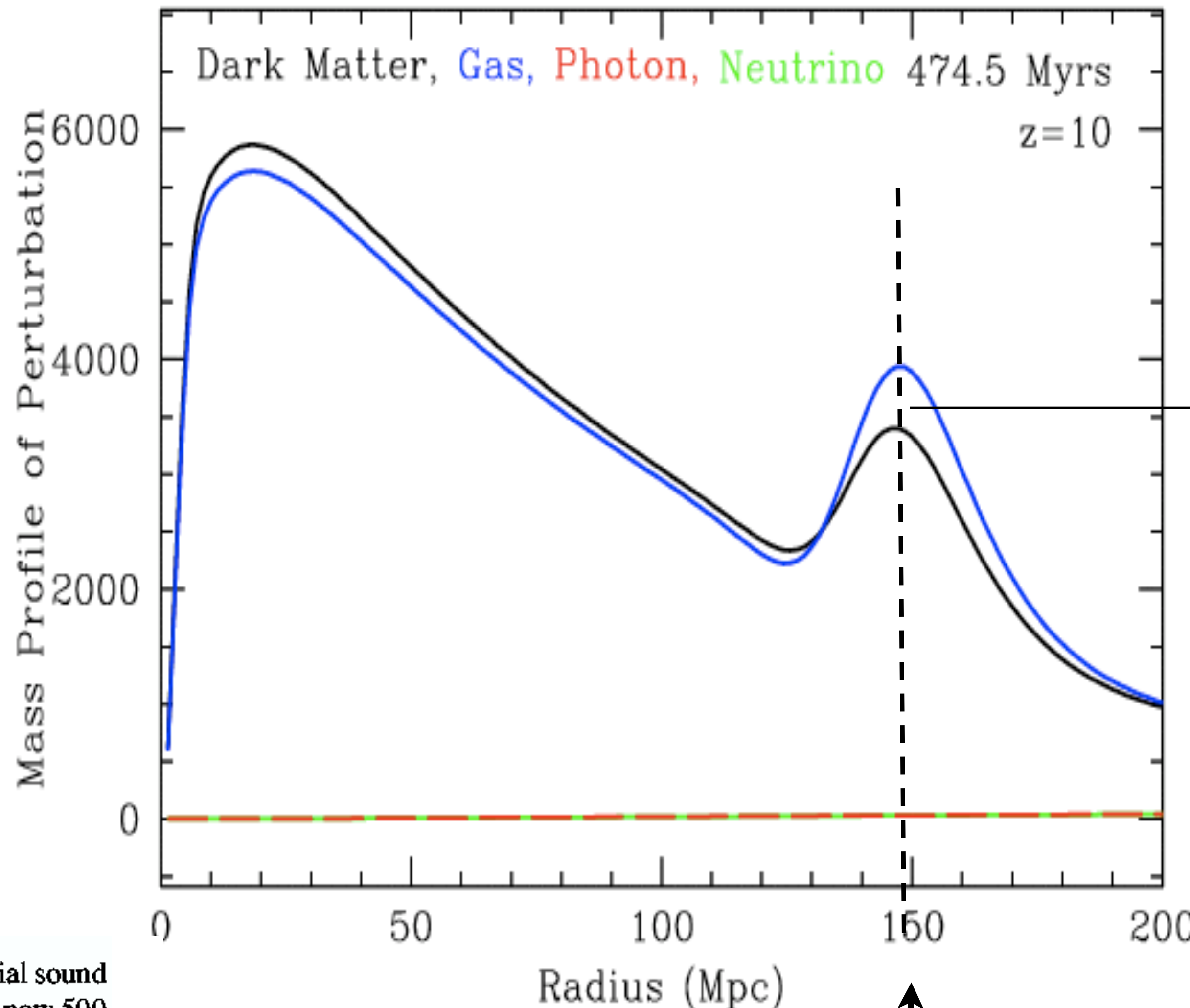
Baryons + DM affect each other after decoupling: How?



Baryons + DM affect each other after decoupling: How?

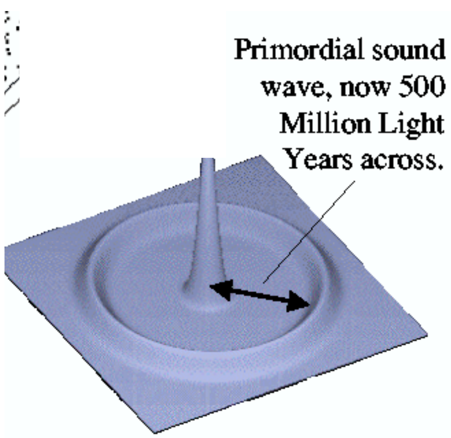


Baryons + DM affect each other after decoupling: How?



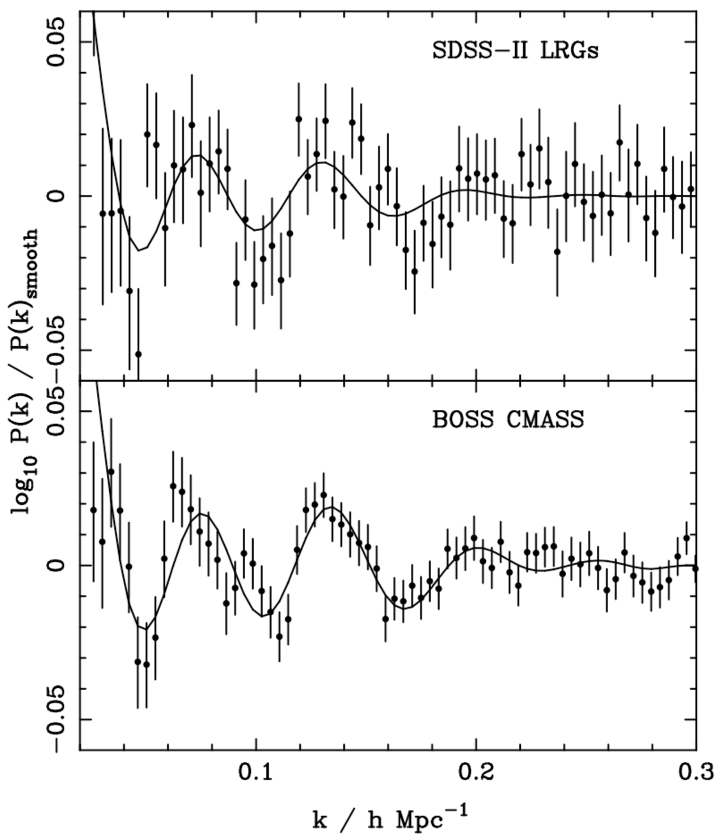
Sound horizon at matter-radiation decoupling

this bump is at 150 Mpc!



Dark matter fluctuations

After decoupling the baryons start following the gravitational potential defined by the dark matter but they retain some of the imprint of the sound-waves at decoupling: the baryon accoustic oscillations.



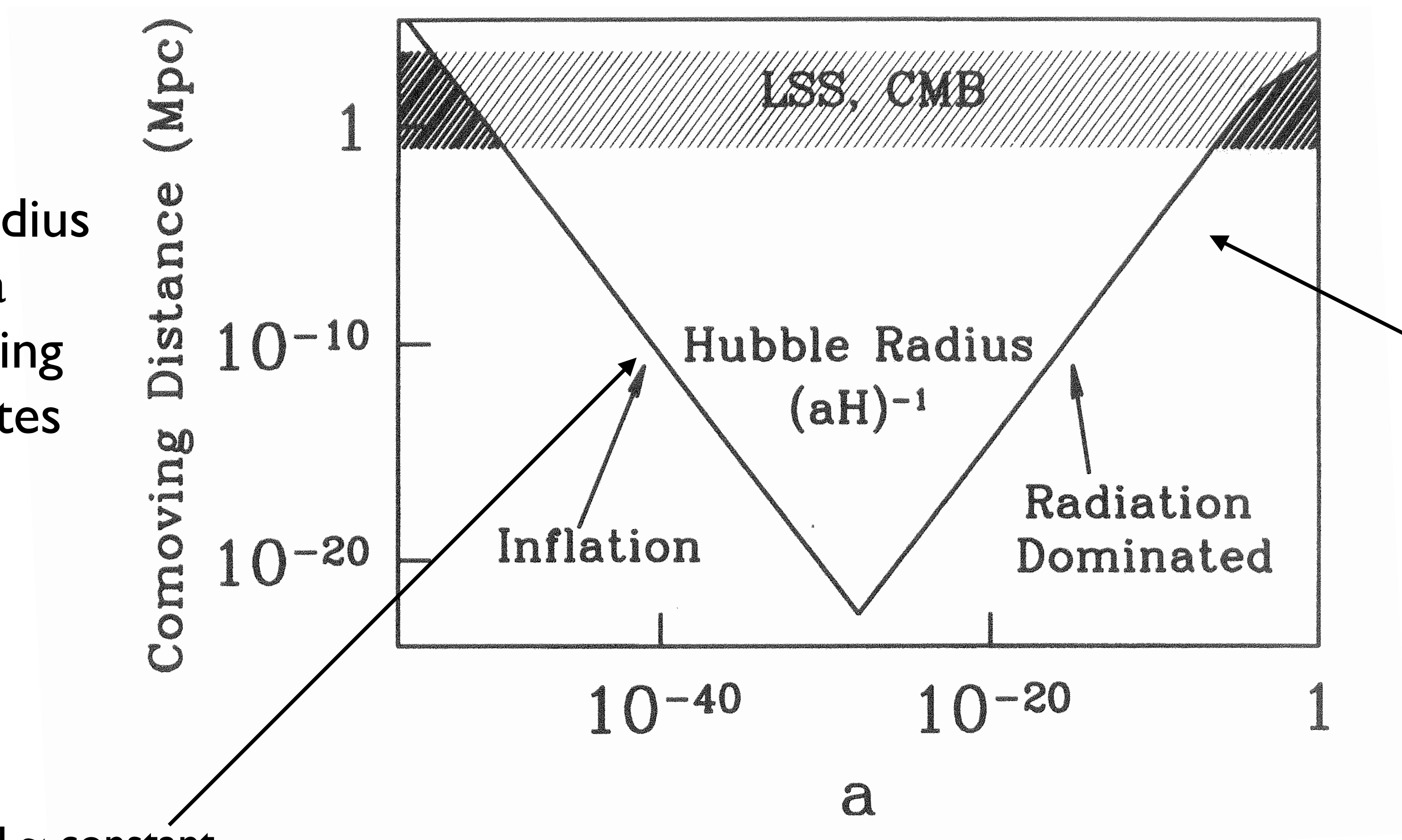
Anderson et al. (2012)

What is the origin of fluctuations in the energy density in the first place?

Remember the situation regarding Horizons and Inflation:

Inflation

Hubble Radius
(c/H)/a
in comoving
coordinates

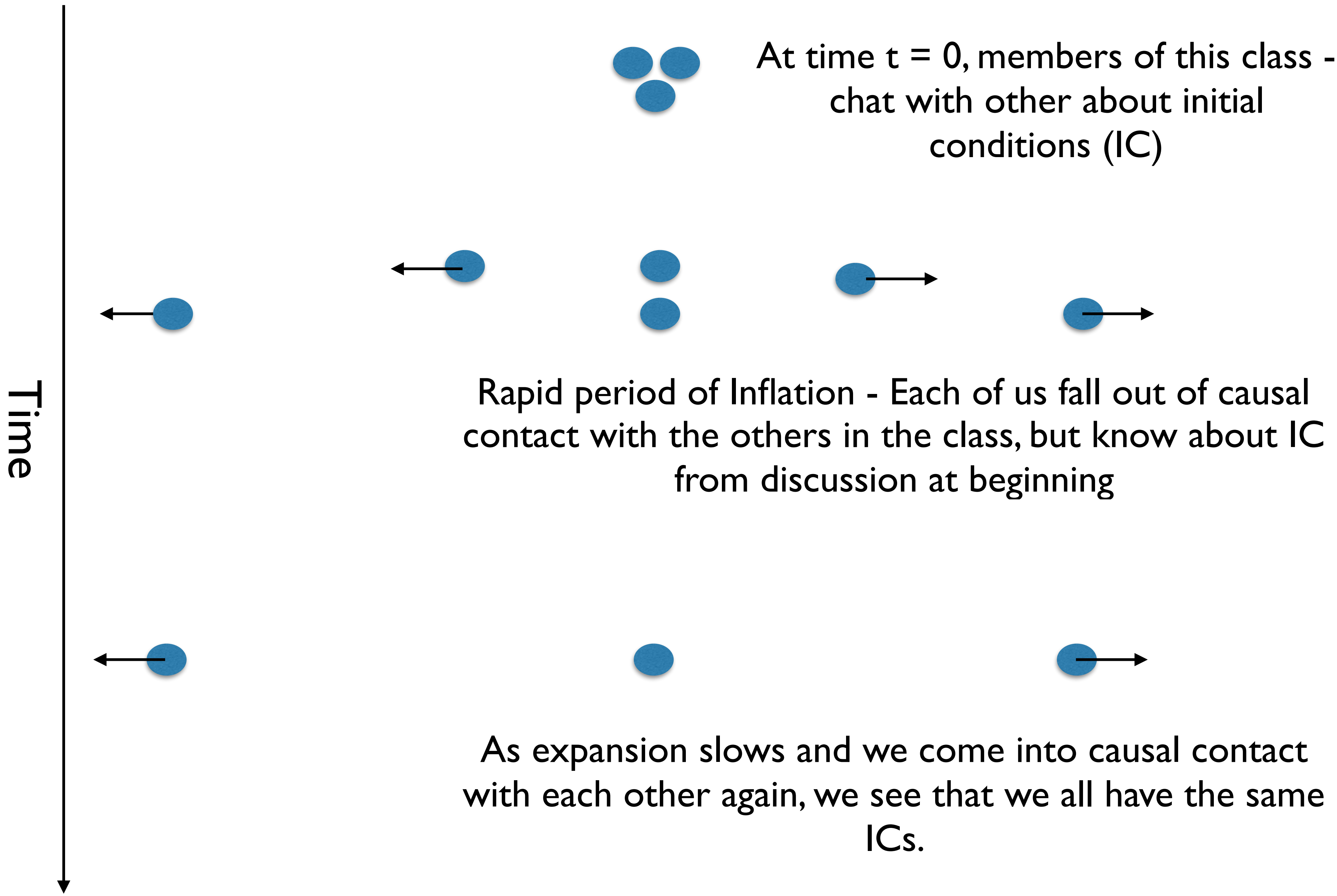


scales as a

H ~ constant
scales as 1/a
$a \propto e^{(\Lambda/3)t}$

Dodelson (2003)

Let's take as an apology the following situation:



Setting Up Primordial Power Spectrum From Inflation

Under non-expanding circumstances, quantum fluctuations die out quickly, but during inflation the expansion is so fast that any fluctuation is moved outside the horizon of any compensating fluctuation. By the time they are back in each other horizon they are in back in each other's horizon, they are no longer quantum scale:

$$a(t) = a(t_{infl})\, e^{H(t-t_{infl})}$$

when inflation started

How long does it take for a quantum fluctuation of size λ_{quant} to freeze out? i.e., what is $\Delta t \sim t - t_{infl}$?

It is given by the time it takes for the fluctuation to expand to the Hubble radius

$$(a_{freeze}/a_{quant})\lambda_{quant} = r_H = c/H$$

$$\Rightarrow \Delta t = (1/H)\ln(a_{freeze}/a_{quant}) = (1/H)\ln(c/H\lambda_{quant})$$

During inflation, $H \sim$ constant and we can reasonably assume the same for λ_{quant}, Δt is constant.

If during inflation, perturbations are generated at a given rate => fixed number per logarithmic interval in space (because of exponential expansion). This continues for many e-folding times and during each interval, the fluctuation looks the same => i.e. power spectrum must be scale free => power law $P(k) \propto k^n$

Power Spectrum

The power spectrum is defined as

$$<\delta(k)\delta^*(k)> = (2\pi)^3\, \delta(k\text{-}k)P(k)$$

isotropy implies that P(k) can only depend on |k|

If δ is a Gaussian field (as predicted by many theorem) then P(k) completely specifies the statistical properties.

P(k) quantifies the amount of clustering for each k-mode.

Setting Up Primordial Power Spectrum From Inflation

If the scalar field that is perturbed is related to the gravitational potential Φ and the fluctuations are of the same amplitude => Δ_Φ^2 = constant

$\Delta_\Phi^2 \equiv k^3 P_\Phi(k) \propto$ constant

but $\nabla^2\Phi = 4\pi G\rho\delta a^2$

$\xRightarrow{\text{Fourier Transform}} k^2\Phi \propto -\delta(k)$

$k^4 P_\Phi(k) \propto P_\delta(k)$ using the definition of P(k)

$P_\delta(k) \propto k^4 P_\Phi(k) \propto k(k^3 P_\Phi(k)) \propto k\Delta_\Phi^2 \propto k$

=> n=1 assuming $P_\delta(k) \propto k^n$

How does structure grow?

	above horizon	below horizon	z
radiation dominated epoch	$\delta \propto a^2$ $P \propto a^4$	no growth	3500
matter dominated epoch	$\delta \propto a$ $P \propto a^2$	$\delta_{DM} \propto a$, $P_{DM} \propto a^2$, oscillations in baryons	1100
after decoupling (if $\Omega_m \sim 1$)	$\delta \propto a$ $P \propto a^2$	$\delta \propto a$ $P \propto a^2$	

Different Growth of Structure on Small + Large Scales

The observed power spectrum is quite different than the primordial power spectrum

$$P(k) = Ak^n T^2(k)$$

Transfer function

T(k) captures the growth of fluctuations in and outside the horizon.

Inflation sends perturbations beyond the horizon, but after the end of inflation the horizon is expanding again.

Perturbations that have not yet entered the horizon continue to grow (to demonstrate this requires a rigorous GR treatment); we saw that $\delta(k)$ grows as a^2 => P(k) gives as a^4

Large scale modes enter later and thus have had more time to grow, but if a mode enters the horizon during radiation domination its growth will cease and instead oscillate due to the radiation pressure.

This does not apply to the DM: during radiation domination it is the radiation fluid that produces the growth of modes, but the DM interacts only through gravity. The density of DM can stream into the gravitational well produced by a perturbation in the radiation fluid, but on small scales this averages out.

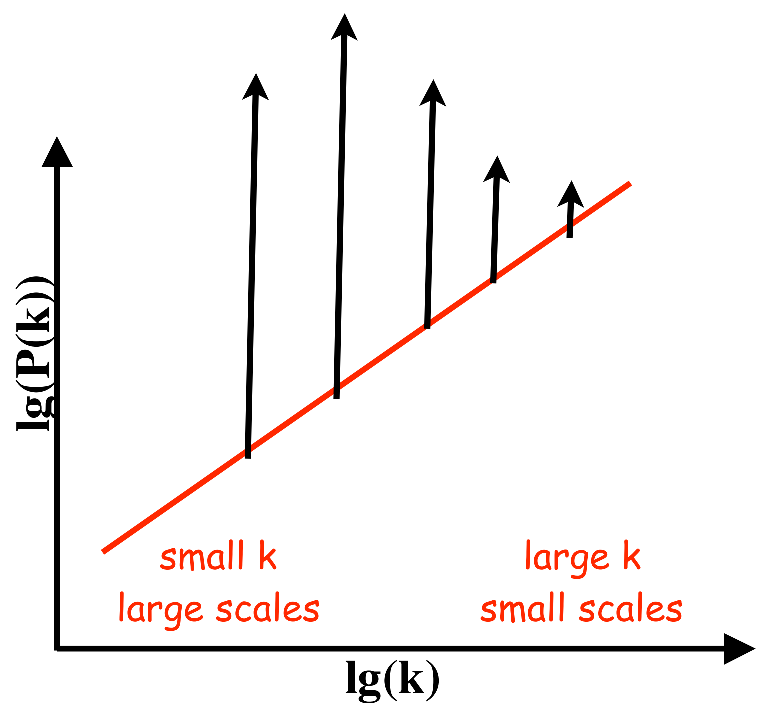
DM growth essentially stalls.

Illustrating why there is a peak in the matter power spectrum

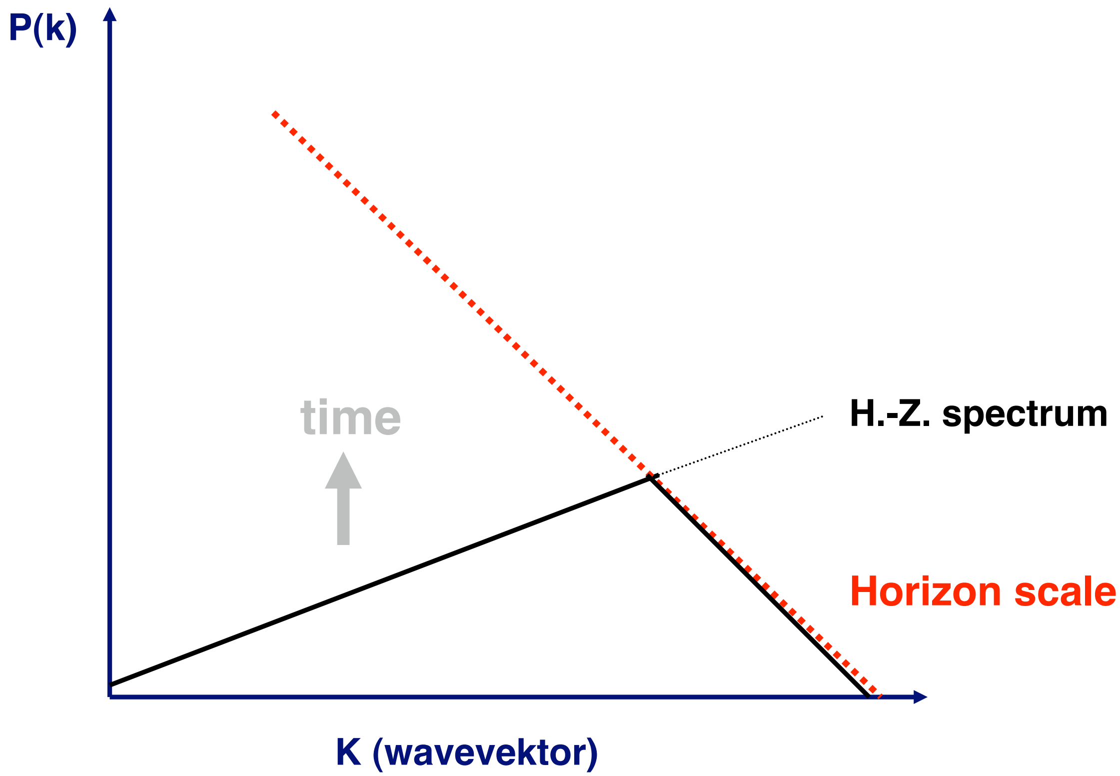
The initial power spectrum of fluctuations is the following:

$$P_0(k) = A\, k^{n_s}$$

Therefore we could expect P(k) at large scales to grow much more than at small scales



Evolution of the Matter Power Spectrum

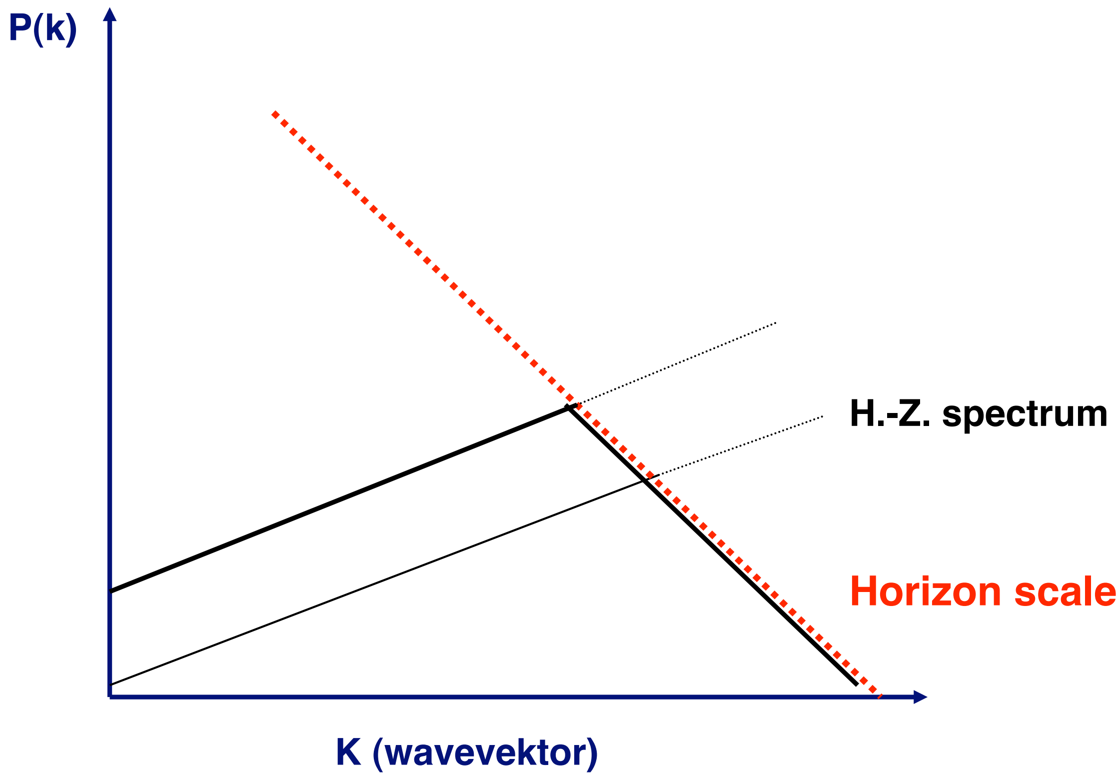


large scales
small k

small scales
large k

Credit: Bohringer

Evolution of the Matter Power Spectrum

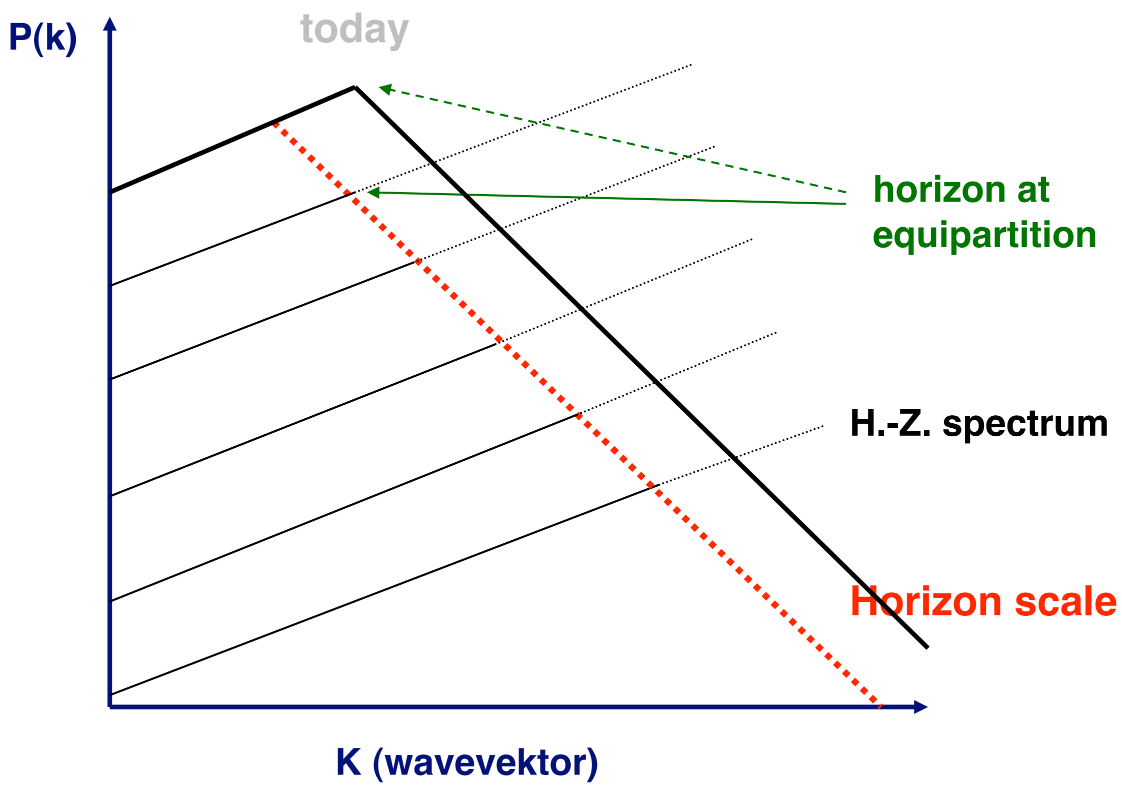


large scales
small k

small scales
large k

Credit: Bohringer

Evolution of the Matter Power Spectrum



large scales
small k

small scales
large k

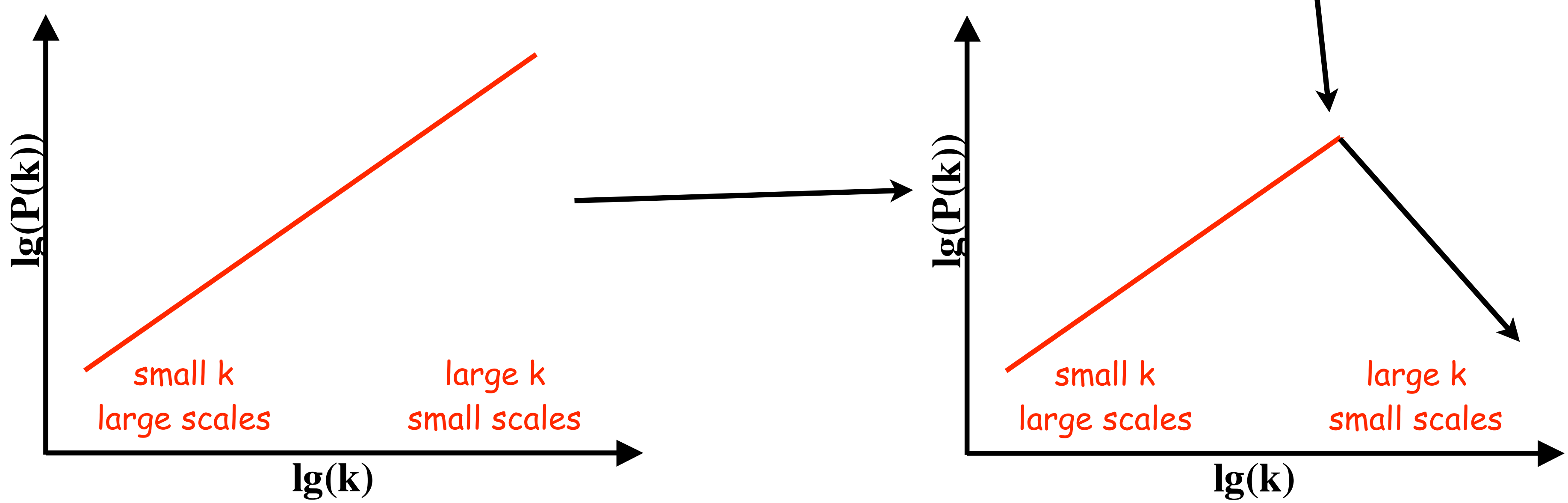
Credit: Bohringer

Illustrating why there is a peak in the matter power spectrum

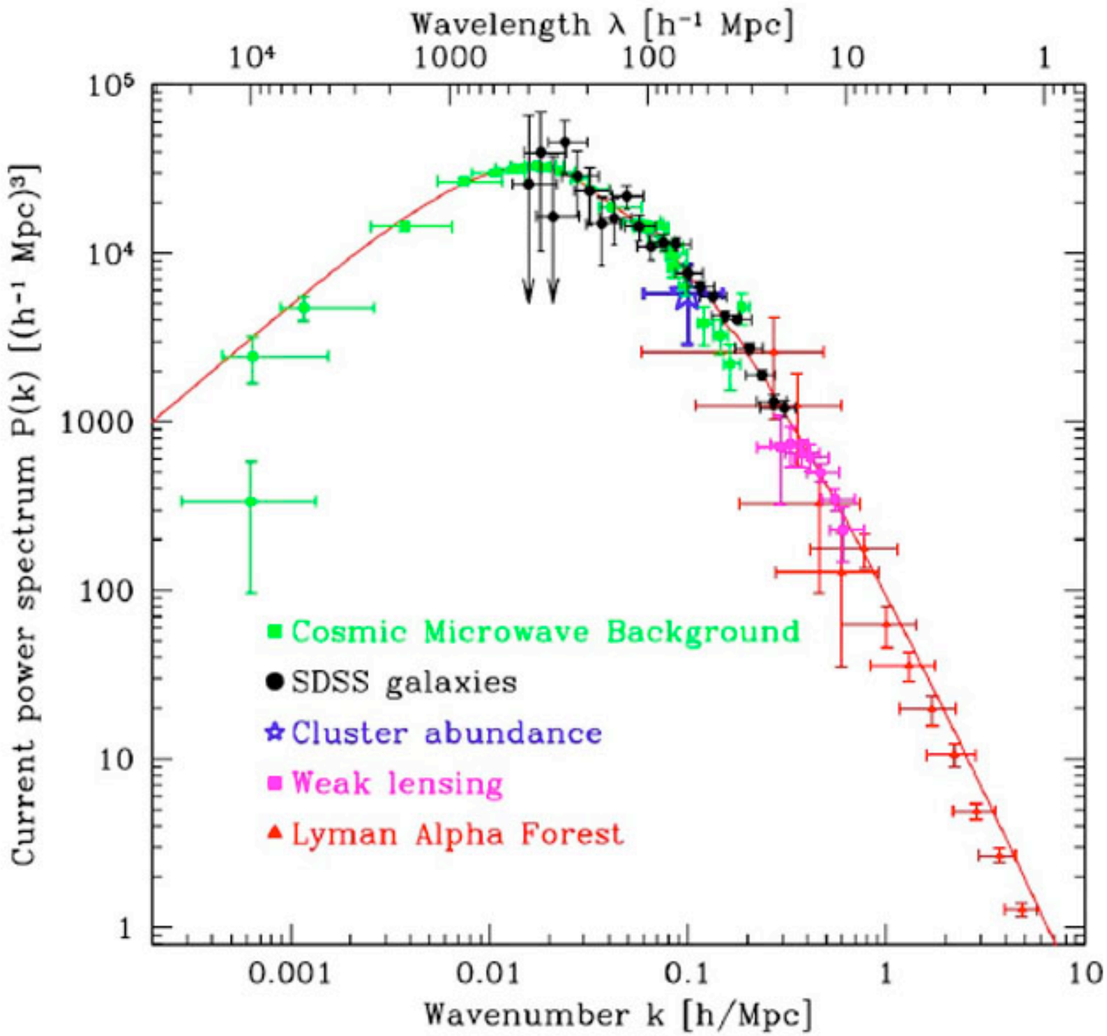
The initial power spectrum of fluctuations is the following:

$$P_0(k) = A\,k^{n_s}$$

Position of turn-over determined by horizon size @ matter-radiation equality



Statistics of matter fluctuation



From Tegmark (2004)

Setting Up Primordial Power Spectrum From Inflation

What is the power spectrum which results?   Let's quantify issues:

Modes entering after a_{eq} have $P(k) \propto k$ and have grown by $(a_{eq}/a_i)^4$

$$P_{eq}(k) = (a_{eq}/a_i)^4 P_i(k) \quad k << k_{enter\text{-}eq}$$

Modes that enter before a_{eq} grew by factor $(a_{enter}(k)/a_i)^4$ where $a_{enter}(k) \propto 1/k$

$$\Rightarrow P_{eq}(k) \propto k^{-3} \quad \text{for } k >> k_{enter\text{-}eq}$$

The power spectrum peak around $k \sim k_{enter\text{-}eq}$

The position of the peak of the power spectrum depends on the Horizon Size

$$d_H = 2c/H = 2c/H_0 (\Omega_{m,0})^{0.5} (1+z)^{-1.5}$$

which is equal to the above in a matter-dominated universe at $z >> 1/\Omega_{m,0}$

A feature of some some length l grows in proportion to a, but as the horizon grows as $d_H \propto a^{3/2}$, so larger features come into causal contact with each other at later times

=> we therefore expect the transfer function to depend on $\Omega_m h^2$ and k

Where is the peak in the power spectrum?

We already found that for $y \equiv \rho_m/\rho_r = a/a_{eq}$ that $\delta_m \propto 1 + (3/2)y$

=> $\delta_m \propto$ constant for $a \ll a_{eq}$ and $\delta_m \propto a$ for $a \gg a_{eq}$

In the radiation-dominated era perturbation modes with $l < d_H(z_{eq})$ enter the horizon but δ is constant.

In the matter-dominated era modes with $l > d_H(z_{eq})$ enter and $\delta \propto a$ and thus δ grows.

=> the power spectrum must have a break on the length scale of the horizon at matter radiation equality:

$d_H(z_{eq}) \sim 16/(\Omega_{m,0}h^2)$ Mpc $\qquad\qquad k \sim 0.06\ \Omega_{m,0}h^2\ \mathrm{Mpc}^{-1}$

For $k < k_{eq}$ (large scales) fluctuations enter the horizon during the matter dominated era and grow as a preserving the initial power spectrum $P(k) \propto k$

For $k > k_{eq}$ (small scales) fluctuations enter the horizon during the radiation dominated era and cannot grow => $P(k) \propto k^{-3}$

Transfer functions

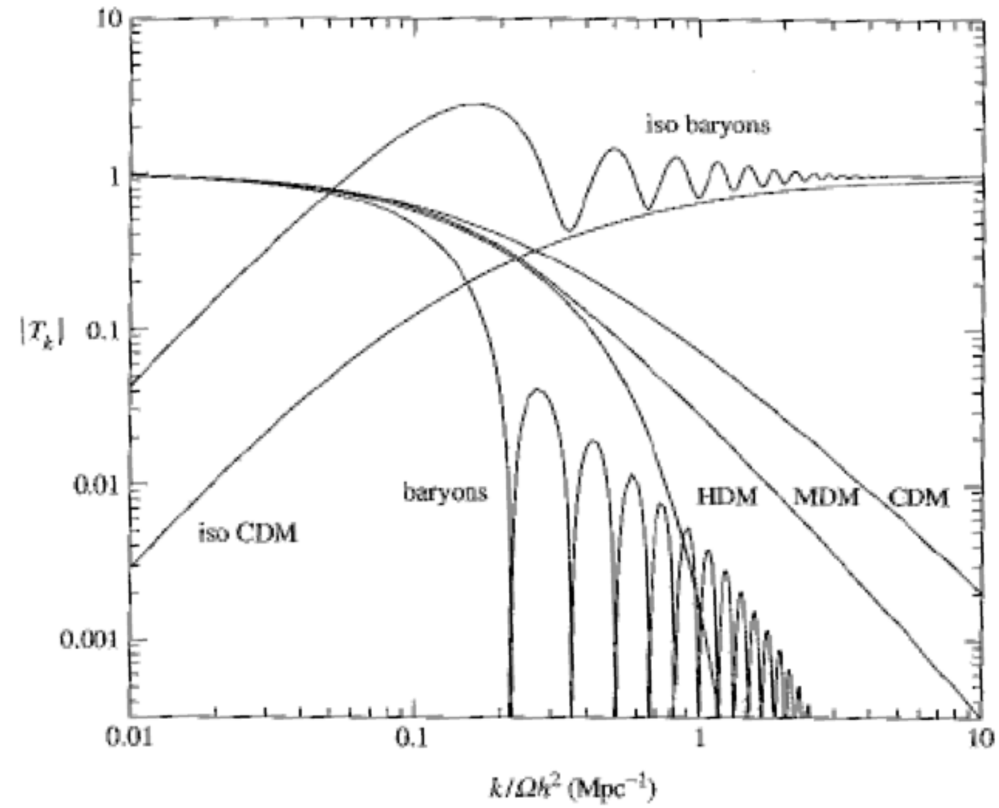


Figure 15.1 Examples of adiabatic transfer functions for baryons, hot dark matter (HDM), cold dark matter (CDM) and mixed dark matter (MDM; also known as CHDM). Isocurvature modes are also shown. Picture courtesy of John Peacock.

Growth of Structure after T_{eq}

After the matter-radiation equality, the power spectrum grows as $P(k) \propto \delta^2 \propto a^2$. The dependence of k_{eq} can be used to constrain Ω_m

$\Delta^2(k) \propto k^4$ for small k and $\Delta^2(k) \propto k^0$ for large k

=> hierarchical structure formation where smaller over densities go non-linear first and collapse earlier.

The baryon acoustic oscillations are superimposed on the dark-matter fluctuations.

Adiabatic Perturbations

In what form do fluctuations in the radiation/matter energy density take? How large are fluctuations in matter relative to fluctuations in radiation and also in the temperature?

Before recombination, the baryons and the radiation were tightly coupled. The entropy per unit mass in a volume has a very high value because of the large value of σ_{rad} (entropy per baryon).

We discussed earlier that the value of σ_{rad} might be related to the microscopic physics of a GUT or electroweak phase transition; if that is correct, then we expect fluctuations to have the same value for σ_{rad} => we expect adiabatic perturbations.

=> entropy is carried almost entirely by radiation

$$S = (4/3)\sigma T^3 V \propto \sigma_{rad} \propto T^3/\rho_m \propto \rho_r^{3/4}/\rho_m \qquad \sigma_{rad} = 4 m_p \sigma_r T^3 / 3 k_B \rho_m$$

An adiabatic perturbation leaves S invariant and consists of fluctuations in both ρ_m and ρ_r such that

$$\delta S/S = 0 = \delta\sigma_{rad}/\sigma_{rad} = (3/4)\ \delta\rho_r/\rho_r - \delta\rho_m/\rho_m = 3\ \delta T/T - \delta\rho_m/\rho_m$$

$$\delta_m \equiv \delta\rho_m/\rho_m = 3\ \delta T/T = (3/4)\ \delta\rho_r/\rho_r \equiv 3/4\ \delta_r$$

Measuring the Matter Power Spectrum From Galaxies: Correlation Function

The two-point correlation function gives the excess probability of finding pairs of objects at a separation r. It is defined as $\xi(r) = \langle\delta(x_1)\delta(x_2)\rangle$ and this is related to the power spectrum through its Fourier Transform

$$\xi(r) = \langle\delta(x_1)\delta(x_2)\rangle = \int d^3k/(2\pi)^3\, e^{ik \cdot x}\, P(k)$$

The power spectrum has units of lengths and it is convenient to define a dimensionless version:

$$\Delta^2(k) = (4\pi k^3\, P(k))/(2\pi)^3$$

The primordial power spectrum is $P(k) = Ak^n$;

if n=1 the model in the Harrison-Zeldovich spectrum, where fluctuations are scale-invariant in the gravitational potential Φ