

Chapter 4

The Hubble Constant

The Hubble constant, $\dot{a}/a \equiv H(a)^1$, is one of the key parameters describing the Universe. It is largely a scaling parameter, setting (along with other parameters to a lesser extent) the age of the Universe, and the absolute values of luminosities and sizes.

There are a number of ways of measuring Hubble's constant, which we will skim in this lecture. The first, and so far most important method, uses the distance–recession velocity relation to estimate Hubble's constant. Other methods are then discussed, including gravitational lensing and distances from cluster Sunayev-Zel'dovich effect measurements. Then the results are summarised.

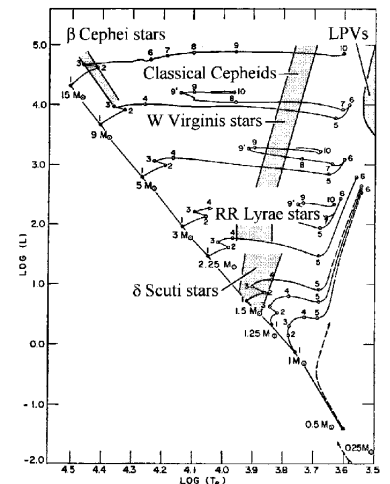
4.1 Hubble constant from Hubble's law

4.1.1 Hubble's law: history

- Vesto Slipher discovered in the 1920s that distant galaxies typically move away from us. Edwin Hubble found that their recession velocity grows with distance,

$$v = H_0 D, \quad (4.1)$$

and determined the constant of proportionality as $H_0 \approx 570 \text{ km s}^{-1} \text{ Mpc}^{-1}$. This value of the Hubble constant is very high, primarily because the absolute magnitude of the Cepheid variable stars had been dramatically overestimated (i.e., the luminosity was underestimated; they were mixed up with W Virginis stars, which show similar variability but dramatically fainter luminosities).



A luminosity–effective temperature relation, with the shaded areas denoting the instability strip.

¹Only astronomers would call something that is manifestly not constant a constant.

- we had seen in (2.17) before that all distance measures in a Friedmann-Lemaître universe follow the linear relation

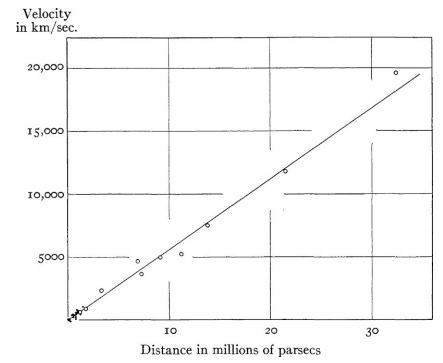
$$D = \frac{cz}{H_0} \quad (4.2)$$

to first order in $z \ll 1$; since $cz = v$ is the velocity according to the linearised relation for the Doppler shift,

$$1 + z = \frac{c + v}{c - v} \approx 1 + \frac{v}{c}, \quad (4.3)$$

(4.2) is exactly the relation that Hubble found;

- there is little doubt that (4.1) is the result that Hubble wanted to find because he wanted his measurements to support the Friedmann-Lemaître cosmology; he even left out data points from the analysis that did not support his conclusion;



The relation between recession velocity and distance originally published by Hubble and Humason in 1931. Note the value of the Hubble constant!

4.1.2 Hubble's law: the challenge

- There are multiple problems with measuring the Hubble constant. Firstly, we have no way (yet) of measuring a direct (trigonometric parallax) measurement to (any) galaxy, making a trigonometric distance impossible to measure (except for masers; see later).
- Thus, a sequence of distance indicators must be used to estimate the distances to (typically) star clusters; these star clusters are used to calibrate other distance indicators which in turn are used to calibrate other distance indicators which give a value for the Hubble constant.
- A second problem is that while (4.2) holds for an idealised, homogeneous and isotropic universe, real galaxies have peculiar motions on top of their Hubble velocity which are caused by the attraction from local density inhomogeneities; for instance, galaxies in our neighbourhood feel the gravitational pull of a cosmologically nearby supercluster called the Great Attractor and accelerate towards it; the galaxy M 31 in Andromeda and the Milky Way approach each other at $\sim 100 \text{ km s}^{-1}$;
- thus, the peculiar velocities of the galaxies must either be known well enough, for which a model for the velocity field is necessary, or they must be observed at so large distances that any peculiar motion is unimportant compared to their Hubble velocity; requiring that peculiar velocities of order $300 \dots 600 \text{ km s}^{-1}$ be less than 10% of the Hubble velocity, galaxies with redshifts

$$z \gtrsim 10 \times \frac{300 \dots 600 \text{ km s}^{-1}}{c} \approx 0.01 \dots 0.02 \quad (4.4)$$

must be observed; this is already so distant that it is hard or impossible to apply direct distant estimators;

- this illustrates why accurate measurements of the Hubble constant are so difficult: nearby galaxies, whose distances are more accurately measurable, do not follow the Hubble expansion well, but the distances to galaxies far enough to follow the Hubble law are very hard to measure;

4.1.3 The distance ladder: the first 20Mpc

- measurements of the Hubble constant from Hubble's law thus require accurate distance measurements out to cosmologically relevant distance scales; since this is impossible in one step, the so-called distance ladder must be applied, in which each step in the ladder calibrates the next;
- There are a number of key steps in the distance ladder.
- **Trigonometric parallaxes:** a key direct distance measurement is the trigonometric parallax caused by the annual motion of the Earth around the Sun; by definition, a star at a distance of a parsec perpendicular to the Earth's orbital plane has a parallax of an arc second; astrometric measurement accuracies of order $10^{-3}''$ are thus necessary to measure distances of order 100 pc at 10σ ;
- in such a way, the distances to local stars and star clusters (notably the Hyades) have been measured (most accurately by the European satellite Hipparcos; see <http://www.rssd.esa.int/index.php?project=HIPPARCOS&page=Hyades> for some information on the Hyades from Hipparcos, and <http://www.rssd.esa.int/index.php?project=HIPPARCOS&page=index> for more general information on Hipparcos), allowing measurement of the absolute magnitude of main sequence stars: this in turn allows the distances to much more distant clusters of stars to be measured.
- **Cluster distances from convergent points:** The distance to nearby clusters can be worked out from their proper motions on the sky and their radial velocities. Just as parallel train tracks (or the edges of roads, or meteors) appear to converge towards a point, so do the paths of stars in a star cluster. The distance to a cluster can be worked out by using the angle to the convergent point θ and the radial velocity v_r and the proper motion μ :

$$D(\text{pc}) = \frac{v_r(\text{km/s}) \tan \theta}{4.74\mu(''/\text{yr})}. \quad (4.5)$$

Please see <http://www.astro.washington.edu/labs/clearinghouse/labs/Hyades/disthyad.html> for a great discussion of this point at length and an example that you can work through.

- **Distances from variable stars:** Star clusters allow access to one of the key distance indicators: *Cepheid variable stars*. Cepheids are high-mass stars in late evolutionary stages which undergo periodic variability (RR Lyraes are also critical distance indicators for old populations); the underlying instability is driven by the temperature dependence of the atmospheric opacity in these stars, which is caused by the transition between singly and doubly ionised Helium;
- the cosmologically important aspect of the Cepheids is that their variability period τ and their luminosity L are related,

$$L \propto \tau^{1.3} , \quad (4.6)$$

hence their luminosity can be inferred from their period, and their distance from the ratio of their luminosity to the flux S observed from them,

$$D = \sqrt{\frac{L}{S}} ; \quad (4.7)$$

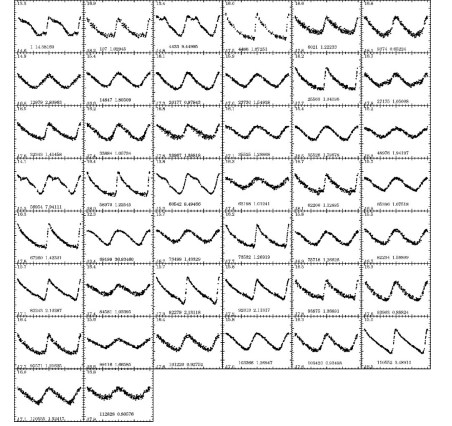
at the relevant distances, any distinction between differently defined distance measures is irrelevant;

- it is of crucial importance here that the calibration of the period-luminosity relation depends on the metallicity of the Cepheids, and thus on the stellar population they belong to; Hubble's originally much too high result for H_0 was corrected when Baade realised that stars in the Galactic disk belong to another stellar population than in the halo;
- by measuring the periods of Cepheids and comparing their apparent brightnesses in star clusters of known distance (from typically main sequence fitting distances, which in turn were calibrated using very nearby clusters with trigonometric distances) and Cepheids in the LMC it was possible to determine the distance to the Large Magellanic Cloud as $D_{\text{LMC}} = (50.1 \pm 3) \text{ kpc}$;
- measuring the periods of Cepheids in the LMC and comparing their apparent brightnesses in different galaxies, it is thus possible to determine the relative distances to the galaxies; for example, comparisons between Cepheids in the LMC and the Andromeda galaxy M 31 show

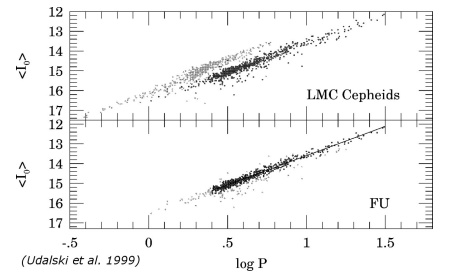
$$\frac{D_{\text{M 31}}}{D_{\text{LMC}}} = 15.28 \pm 0.75 , \quad (4.8)$$

while Cepheids in the member galaxies of the Virgo cluster yield

$$\frac{D_{\text{Virgo}}}{D_{\text{LMC}}} = 316 \pm 25 ; \quad (4.9)$$



Some examples for Cepheid lightcurves



In some Cepheids, overtones of the pulsation are excited rather than the fundamental mode.

- of course, for the Cepheid method to be applicable, it must be possible to resolve at least the outer parts of distant galaxies into individual stars and to reliably identify Cepheids among them; this was one reason why the Hubble Space Telescope was proposed, to apply the superb resolution of an orbiting telescope to the measurement of H_0 ; Cepheid distance measurements are possible to distances $\lesssim 20$ Mpc;
- **Masers:** There is a critical consistency check with the trigonometric parallax/main sequence fitting/Cepheid distance scale: water masers orbiting the central regions of distant galaxies. The principle is simple: if there is perfect circular motion in the maser ring and the inclination is known and not face on, then the proper motion of the maser clouds can be combined with the radial velocity of the clouds at the orbit tangent points to provide a distance:

$$D(\text{Mpc}) = \frac{v_r(\text{km/s})}{4.74 \cos(90 - i) \mu(\mu\text{as/yr})}. \quad (4.10)$$

In practice, the acceleration of the masers can also be measured using radio interferometric observations with VLBI. The disk rotation of ~ 1000 km/s coupled with $30 \mu\text{as/yr}$ gives a distance of 7.3 ± 0.3 Mpc; a direct geometric distance to a galaxy with Cepheids.

- **Eclipsing binary stars:** Eclipsing binaries provide an accurate method of measuring distances to nearby galaxies with an accuracy of 5%. A review of the method can be found from Paczynski (1997). The method requires both photometry and spectroscopy of an eclipsing binary. From the light and radial velocity curve the fundamental parameters of the stars can be determined accurately. The light curve provides the fractional radii of the stars, which are then combined with the spectroscopy to yield the physical radii and effective temperatures. The velocity semi-amplitudes determine both the mass ratio and the sum of the masses, thus the individual masses can be solved for. Furthermore, by fitting synthetic spectra to the observed ones, one can infer the effective temperature, surface gravity and luminosity. Comparison of the luminosity of the stars and their observed brightness yields the reddening of the system and distance. Measuring distances with eclipsing binaries is an essentially geometric method and thus accurate and independent of any intermediate calibration steps. With the advent of 8 m class telescopes, eclipsing binaries have been used to obtain accurate distance estimates to the LMC, SMC, M31 and M33.
- **Light echos:** With SN1987A, it was possible to measure the distance to the LMC using a light echo. The idea is that if one has a spherical shell or ring which existed before the supernova (ring

in this case), then it lights up after a certain time when hit by the light from SN1987A (the front side), and will remain illuminated until the echo has faded from the back side. Given an angular diameter at that time from direct measurements, the distance to the LMC can be measured.

4.1.4 Distance Ladder: extending beyond 20Mpc

- **Fundamental plane:** scaling relations within classes of galaxies provide additional distance indicators; in the three-dimensional parameter space spanned by the velocity dispersion σ_v , the effective radius R_e and the surface brightness I_e at the effective radius, elliptical galaxies populate the tight *fundamental plane* defined by

$$R_e \propto \sigma_v^{1.4} I_e^{-0.85} ; \quad (4.11)$$

since the luminosity is evidently

$$L \propto I_e R_e^2 , \quad (4.12)$$

the fundamental-plane relation implies

$$L \propto \sigma_v^{2.8} I_e^{-0.7} ; \quad (4.13)$$

such a relation follows directly from the virial theorem if the mass-to-light ratio in elliptical galaxies increases gently with mass,

$$\frac{M}{L} \propto M^{0.2} ; \quad (4.14)$$

- thus, if it is possible to measure the effective surface brightness I_e (which does not depend on distance, if one neglects cosmological surface brightness dimming and k -corrections) and the velocity dispersion σ_v of an elliptical galaxy, the fundamental plane gives the luminosity, which can be compared to the flux to find the distance; such distances are accurate to 11% in the best cases (i.e., 22% intrinsic scatter in luminosity).
- **Tully-Fisher relation:** a relation similar to (4.13), the Tully-Fisher relation, holds for spiral galaxies if the velocity dispersion σ_v is replaced by the rotational velocity v_{rot} and if surface brightness is neglected; however, spiral galaxies avoid galaxy clusters, and it is therefore more difficult to decide whether they belong to a galaxy cluster such as Virgo or Coma;
- The form of the relation is $L \propto v_{\text{rot}}^\alpha$ where α is between 2.5 (in blue bands) and 4 (in the near-infrared). The scatter in the Tully-Fisher relation can be as little as 0.2 mags or less in carefully-selected samples in the far red and near-infrared.

- **Surface Brightness Fluctuations:** an interesting way for determining distances to galaxies uses the fluctuations in their surface brightness; the idea behind this method is that the fluctuations in the surface brightness will be dominated by the rare brightest stars, and that the optical luminosity of the entire galaxy will be proportional to the number N of such stars; assuming Poisson statistics, the fluctuation level will be proportional to \sqrt{N} , from which N and $L \propto N$ can be determined once the method has been calibrated with galaxies whose distance is known otherwise; again, the distance is then found by comparing the flux to the luminosity;
- **Planetary Nebulae:** Planetary nebulae, which are late stages in the evolution of stars, have a luminosity function with a steep upper cut-off; moreover, their spectra are dominated by sharp nebular emission lines which facilitate their detection even at large distances because they appear as bright objects in narrow-band filters tuned to the emission lines; since the cut-off luminosity is known, it can be converted to a distance as usual;
- **Supernova Type Ia:** One of the most important classes of distance indicators are supernovae of type Ia; they occur in binary systems in which one of the components is a white dwarf accreting mass from an overflowing companion; since the electron degeneracy pressure in the cores of white dwarfs can stabilise them only up to the Chandrasekhar mass of $\approx 1.4 M_{\odot}$, the white dwarf suddenly collapses once mass accretion drives it over this limit; in the ensuing supernova explosion, part of the white dwarf's material is converted to elements of the iron group; since the amount of nuclear fuel is fixed by the Chandrasekhar mass, the explosion energy is also fixed, and thus so is the luminosity;
- this idealised picture needs to be modified because the amount of energy released depends on the opacity of the material surrounding the supernovae explosion; this leads to a scatter in the peak luminosities, but this scatter can be corrected applying the empirical *Phillips relation*, which relates the peak luminosity L to the time scale τ of the light-curve decay,

$$L \propto \tau^{1.7} ; \quad (4.15)$$

when this correction is applied, type-Ia supernova are turned into precise standard candles with a dispersion of only 6%;

- Because this is one of the brightest standard candles, it has been applied out to redshifts of 1.5 (with difficulty); it was the first convincing and is still one of the most important observations indicating the accelerating expansion of the Universe (see later for a deeper discussion).

- **Type II supernovae:** although they are not standard (or standardisable) candles, core-collapse supernovae of type II can also be used as distance indicators through the *Baade-Wesselink method*; suppose the spectrum of the supernova photosphere can be approximated by a Planck curve whose temperature can be determined from the spectral lines; then, the Stefan-Boltzmann law says that the total luminosity is

$$L = aR^2T^4, \quad (4.16)$$

where a is again the Stefan-Boltzmann constant from (??); the photospheric radius, however, can be inferred from the expansion velocity of the photosphere, which is measurable by the Doppler shift in the emission lines, times the time after the explosion; when applied to the supernova SN 1987A in the Large Magellanic Cloud, the Baade-Wesselink method yields a distance of

$$D_{\text{LMC}} = (44 \dots 50) \text{ kpc}, \quad (4.17)$$

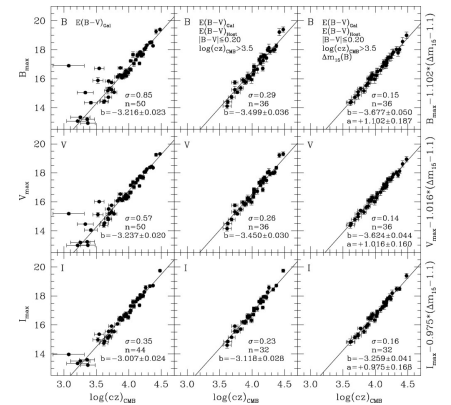
which agrees with other distance measurements (Cepheids, eclipsing binaries, etc).

4.1.5 The HST Key Project

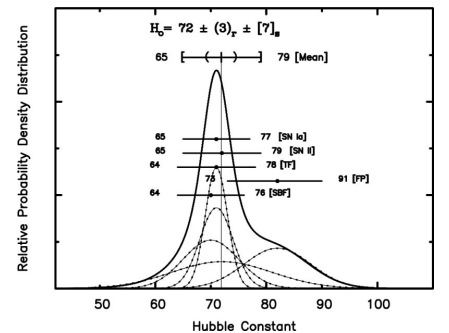
- all these distance indicator were used by the *HST Key Project* to determine accurate distances to 26 galaxies between 3 ... 25 Mpc, and five very nearby galaxies² for testing and calibration;
- double-blind photometry was applied to the identified distance indicators; since Cepheids tend to lie in star-forming regions and are thus attenuated by dust, and since their period-luminosity relation depends on metallicity, respective corrections had to be carefully applied;
- then, the measured velocities had to be corrected by the peculiar velocities, which were estimated by a model for the flow field;
- the estimated luminosities of the distance indicators could then be compared with the extinction-corrected fluxes to determine distances, whose proportionality with the velocities corrected by the peculiar motions finally gave the Hubble constant; a weighted average over all distance indicators is

$$H_0 = (72 \pm 8) \text{ km s}^{-1} \text{ Mpc}^{-1}, \quad (4.18)$$

where the error is the square root of the systematic and statistical errors summed in quadrature;



Hubble laws as measured by the *Hubble Key Project* in different wave bands (top to bottom) and in different stages of correction (left to right).



Probability distributions for H_0 obtained with different measurement techniques applied in the *Hubble Key Project*, and the combined distribution.

²see <http://www.ipac.caltech.edu/H0kp/H0KeyProj.html>

4.2 Gravitational Lensing

- a totally different method for determining the Hubble constant uses gravitational lensing; masses bend passing light paths towards themselves and therefore act in a similar way as convex glass lenses; as in ordinary geometrical optics, this effect can be described applying Fermat's principle to a medium with an index of refraction

$$n = 1 - \frac{2\Phi}{c^2}, \quad (4.19)$$

where Φ is the Newtonian gravitational potential;

- if it is strong enough, the curvature of the light paths causes multiple images to appear from single sources; compared to the straight light paths in absence of the deflecting mass distribution, the curved paths are geometrically longer, and they have to additionally propagate through a medium whose index of refraction is $n > 1$; this gives rise to a *time delay* which has a geometrical and a gravitational component,

$$\tau = \frac{1}{2} (\vec{\theta} - \vec{\beta})^2 - \psi(\vec{\theta}), \quad (4.20)$$

where $\vec{\theta}$ are angular coordinates on the sky and $\vec{\beta}$ is the angular position of the source; ψ is the appropriately scaled Newtonian potential of the deflector, projected along the line-of-sight; according to Fermat's principle, images occur where τ is extremal, i.e. $\vec{\nabla}_{\theta}\tau = 0$;

- the projected *lensing potential* ψ is related to the surface-mass density Σ of the deflector by

$$\vec{\nabla}^2\psi = 2\frac{\Sigma}{\Sigma_{\text{cr}}} \equiv 2\kappa, \quad (4.21)$$

where the *critical surface-mass density*

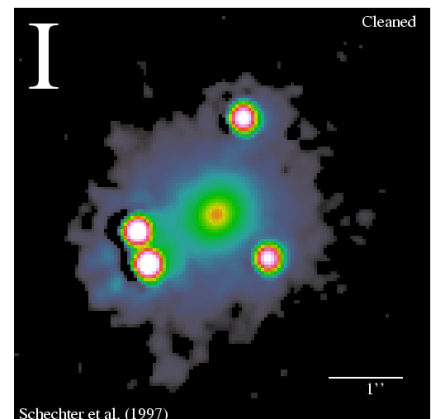
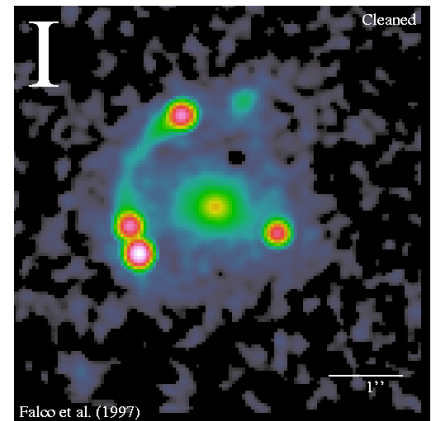
$$\Sigma_{\text{cr}} \equiv \frac{c^2}{4\pi G} \frac{D_s}{D_d D_{\text{ds}}} \quad (4.22)$$

contains the distances $D_{\text{d,s,ds}}$ from the observer to the deflector, the source, and from the deflector to the source, respectively;

- the dimension-less time delay τ from (4.20) is related to the true physical time delay t by

$$t \propto \frac{\tau}{H_0}, \quad (4.23)$$

where the proportionality constant is a dimension-less combination of the distances $D_{\text{d,s,ds}}$ with the Hubble radius cH_0^{-1} and the deflector redshift $1 + z_d$; (4.23) shows that the true time delay is proportional to the Hubble time, as it can intuitively be expected;



The quasars MG 0414 (top) and PG 1115 are quadruply gravita-

- time delays are measurable in multiple images of a variable source; the variable signal arrives after different times in the images seen by the observer, and if the deflector is a galaxy, time delays are typically of order days to months and therefore observable with a reasonable monitoring effort;
- interestingly, it can be shown in an elegant, but lengthy calculation that measured time delays can be inverted to find the Hubble constant from the approximate equation

$$H_0 \approx A(1 - \langle \kappa \rangle) + B\langle \kappa \rangle(\eta - 1), \quad (4.24)$$

where A and B are constants depending on the measured image positions and time delays, $\langle \kappa \rangle$ is the mean scaled surface-mass density of the deflector averaged within an annulus bounded by the image positions, and $\eta \approx 2$ is the logarithmic slope of the deflector's density profile;

- therefore, if a model exists for the gravitationally-lensing galaxy, the Hubble constant can be found from the positions and time delays of the images; applying this technique to five different lens systems³, Kochanek (2002) found

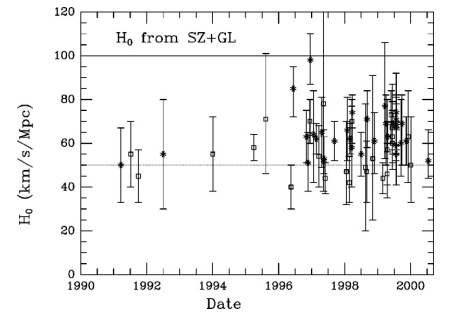
$$H_0 = (73 \pm 8) \text{ km s}^{-1} \quad (4.25)$$

assuming that the lensing galaxies have radially constant mass-to-light ratios;

- this result is highly remarkable because it was obtained in one step without any reference to the extragalactic distance ladder; although there is the remaining ambiguity from the mass model for the lensing galaxies, the perfect agreement between the results from lensing time delays and the HST Key Project is a very reassuring confirmation of the cosmological standard model;

4.3 The Sunyaev-Zel'dovich effect

- another method should finally be mentioned because it is physically interesting and conceptually elegant, although it will probably never become competitive; it is based on two different types of observations of the hot gas in massive galaxy clusters;
- galaxy clusters contain diffuse, fully ionised plasma with temperatures of order (1 ... 10) keV which emits X-rays by the thermal bremsstrahlung (free-free emission) of the electrons scattering off



Values for the Hubble constant obtained with alternative methods (gravitational lensing, GL, and the thermal Sunyaev-Zel'dovich effect, SZ) not depending on the distance ladder.

³these are: PG 1115 + 80, SBS 1520 + 530, B 1600 + 434, PKS 1830 – 211 and HE 2149 – 2745

the ions; as a two-body process, the bremsstrahlung emissivity j_X is proportional to the product of the electron and ion densities n_e and n_i , times the square root of the temperature T ,

$$j_X \propto n_e n_i \sqrt{T} = C_X n_e^2 \sqrt{T} , \quad (4.26)$$

where C_X is a constant whose value is irrelevant for our current purposes; moreover, we have used that the ion density will be proportional to the electron density n_e ;

- since the emissivity is the energy released per volume per time, the energy emitted by the galaxy cluster per surface-area element dA is

$$dE = dA \int dl j_X , \quad (4.27)$$

where the integral extends along the line-of-sight; the energy flux seen by the observer from this surface-area element is

$$dS = \frac{dE}{4\pi D_{\text{lum}}^2} = \frac{dA \int dl j_X}{4\pi D_{\text{lum}}^2} , \quad (4.28)$$

- by definition of the angular-diameter distance, the surface-area element dA spans the solid angle element $d\Omega = dA/D_{\text{ang}}^2$, so the X-ray flux per unit solid angle, or the X-ray surface brightness, is

$$I = \frac{dS}{d\Omega} = \frac{D_{\text{ang}}^2}{4\pi D_{\text{lum}}^2} \int dl j_X = \frac{1}{4\pi(1+z)^4} \int dl j_X , \quad (4.29)$$

where we have used the remarkable *Etherington relation* between the angular-diameter and luminosity distances,

$$D_{\text{lum}} = (1+z)^2 D_{\text{ang}} , \quad (4.30)$$

which holds in any space-time;

- the hot electrons in the galaxy clusters scatter microwave background photons passing by to much higher energies by inverse Compton scattering; this process will neither create nor destroy photons, but transport the photons to higher energy; thus, if the CMB is observed towards a galaxy cluster, its intensity at low photon energies will appear reduced, and increased at high energies; this is the so-called *thermal Sunyaev-Zel'dovich effect*: clusters cast shadows on the CMB at low frequencies, and appear as sources at high frequencies, where the division line lies at 217 GHz;
- the amplitude of the thermal Sunyaev-Zel'dovich effect is quantified by the Compton- y parameter,

$$y = \int dl \frac{kT}{m_e c^2} \sigma_T n_e , \quad (4.31)$$

where m_e is the electron rest-mass and σ_T is the Thomson scattering cross section; the total Compton- y parameter of a galaxy cluster, integrated over the entire solid angle of the cluster, is thus

$$Y = \int d\Omega y = \frac{1}{D_{\text{ang}}^2} \int dA dl y = \frac{1}{D_{\text{ang}}^2} \int dV \frac{kT}{m_e c^2} \sigma_T n_e, \quad (4.32)$$

i.e. it is determined by a volume integral over the cluster divided by the squared angular-diameter distance;

- the comparison between the two observables discussed here, the X-ray surface brightness (4.29) and the integrated Compton- y parameter (4.32), shows that they both depend on the distribution of temperature and electron density within the cluster, and on the squared angular-diameter distance to the cluster; assuming a model for radial T and n_e profiles then allows combining the two types of measurement to find the cluster's angular-diameter distance, which is proportional to the Hubble length cH_0^{-1} and thus to the inverse Hubble constant;
- in this way, it is possible to estimate the Hubble constant by combining X-ray and thermal Sunyaev-Zel'dovich measurements on galaxy clusters; typical values for H_0 derived in this way are substantially lower than the values discussed above, which is probably due to overly simplified assumptions about the temperature and electron-density distributions in the clusters;

4.4 Summary

- if we accept the result of the *Hubble Key Project* for now,

$$H_0 = (72 \pm 8) \text{ km s}^{-1} \text{ Mpc}^{-1}, \quad (4.33)$$

we can calibrate several important numbers that scale with some power of the Hubble constant;

- first, in cgs units, the Hubble constant can be written

$$H_0 = (2.3 \pm 0.3) \times 10^{-18} \text{ s}^{-1}, \quad (4.34)$$

which implies the Hubble time, i.e. the inverse of the Hubble constant

$$\frac{1}{H_0} = (13.6 \pm 1.5) \text{ Gyr} \quad (4.35)$$

and the Hubble radius

$$\frac{c}{H_0} = (1.3 \pm 0.1) \times 10^{28} \text{ cm} = (4.1 \pm 0.5) \text{ Gpc}; \quad (4.36)$$

the critical density of the Universe turns out to be

$$\rho_{\text{cr}} = \frac{3H_0^2}{8\pi G} = (9.65 \pm 2.1) \times 10^{30} \text{ g cm}^{-3} ; \quad (4.37)$$

- the uncertainty in H_0 is conventionally expressed in terms of the dimension-less parameter $h \equiv H_0/100 \text{ km s}^{-1} \text{ Mpc}^{-1}$; since lengths in the Universe are typically measured with respect to the Hubble length, they are often given in units of $h^{-1} \text{ Mpc}$; similarly, luminosities are typically obtained by multiplying fluxes with squared luminosity distances and are thus often given in units of $h^{-2} L_\odot$; we avoid this notation in the following and insert $h = 0.72$ where needed;