Chapter 2

The cosmological standard model

2.1 Introduction

- One of the landmark achievements of the last decade of astronomical study is the establishment of a 'cosmological standard model'. By this term, we mean a consistent theoretical background which is at the same time simple and broad enough to offer coherent explanations for the vast majority of cosmoogical phenomena.
- This lecture will explain and discuss the empirical evidence to which this cosmological standard model owes its convincing power. The construction of homogeneous and isotropic cosmologies from general relativity, and the study of their physical properties and evolution, is treated elsewhere (see, e.g. the separate lecture scripts on general relativity and on cosmology).
- We will start with a brief overview of a few relevant observational facts about the Universe which played a critical role in motivating our current cosmological picture, continue with a short timeline of how it is currently imagined that the Universe evolved, and we will review a few key aspects of the cosmological model. The bulk of the course will discuss in depth a number of observations/methods, and how they fit in to our current cosmological picture.

2.2 Observational overview; the basics

- **Galaxies exist** - the Universe is filled with hundreds of billions of galaxies, with a wide range in properties. In their inner parts, their mass is dominated by cold gas (mostly neutral or molecular hydrogen) and stars (really dense hydrogen!).

- **Exapanding Universe** - Slipher and Hubble demonstrated convincingly in the 20s and 30s that the Universe is expanding, and that the redshift is proportional to distance (at least locally). The truly astounding result, initially determined in the 1990s using Supernova Ia, is that the expansion of the Universe is *accelerating* at the present time. Naïvely, this is counterintuitive, inasmuch as the matter content of the Universe should be decelerating its own expansion.

- **Dark Matter** - both inside galaxies and in galaxy clusters, motions of gas and stars imply gravitational masses dramatically in excess of any mass plausibly associated with stars and gas, i.e., $v \gg \sqrt{GM_{\mathrm{visible}}/R}$. This discrepancy was first noted in the 1940s by Fritz Zwicky, and became really obvious in the 1970s in the study of spiral galaxy rotation curves.

- **Cosmic Microwave Background** - There is a cosmic microwave background with $T \sim 2.73K$. Its main defining feature is its astonishing level of homogeneity – it is flat to 1 part in 10^5.

- **Helium** - There is way too much Helium in the Universe to have been made in just stars (or, put differently, the ratio between Helium and the other, heavier elements is just too high to be explained by being made in stars alone).

There are a large number of other important observations, but these are the key aspects which are useful to bear in mind when trying to parse the elements of our current cosmological picture.

2.3 A brief history of time

Our cosmological picture unifies ‘known physics’ with a few novel ingredients motivated entirely by astronomical observation.

- **Inflation** - it is currently postulated that the Universe had *very* early in its evolution a phase of exponential expansion (roughly 60 e-foldings) which took microscopically small parts of the Universe and boosted them in scale to giant, macroscopic scales ($\sim$ galaxy scales and larger). This has two main advantages. It solves the flatness problem (the Universe would recollapse or expand very rapidly in the case of $\Omega_{\mathrm{initial}} > 1$ or $\Omega_{\mathrm{initial}} < 1$. $\Omega_{\mathrm{initial}}$ needs to be within 1 part in 10^{60} of unity to ensure that $\Omega \sim 1$ today (as

$|\Omega_{\text{total}} - 1| \propto t$, and given that the age of the Universe is $\sim 10^{17}s$ today and evaluating the Ω_{initial} at the Planck time $t_{\text{Planck}} \sim 10^{-43}s$). There is the advantage also that it solves the horizon problem, and blows up tiny quantum fluctuations in the initial density field into density perturbations that later seed structure formation. *Inflation is completely motivated by cosmology*, although physics theorists are happy about it.

- **Radiation Dominated Era** - At this time $T \propto 1/a$; i.e., as the Universe expands it cools also. Protons and neutrons freeze out of the mix as $T \ll 1\,\text{GeV}$. The ratio of p to n is determined by their mass difference coupled with the decay half life of neutrons, ending up in a n/p number ratio of 1/7. When temperatures are low enough $T \sim 1\,\text{MeV}$ nucleosynthesis can proceed, forming ^2H, ^3H, ^4He, ^7Li, etc.

- **Recombination** - At $T \sim 3000$K, the hydrogen can no longer stay ionised, so it recombines. There are no more electrons to Thompson scatter off of, so the Universe suddenly becomes transparent and photons can freely propagate. The CMB is a relic of this transition, and offers a direct view of the structure of the Universe at this time.

- **Matter Dominated Era** - structure formation, galaxy formation, expansion of the Universe. A 'boring' time for cosmologists (who want to measure the formation of structure and the expansion history to measure cosmological parameters), and the most exciting time for astrophysicists (galaxy formation, star formation, planet formation, life, etc - minor details!) The main astrophysically-motivated ingredients here are *dark matter* and *dark energy / cosmological constant*.

2.4 Friedmann models

2.4.1 The metric

- Cosmology deals with the physical properties of the Universe as a whole. The only of the four known interactions which can play a role on cosmic length scales is gravity. Electromagnetism, the only other interaction with infinite range, has sources of opposite charge which tend to shield each other on comparatively very small scales. Cosmic magnetic fields can perhaps reach coherence lengths on the order of $\gtrsim 10\,\text{Mpc}$, but their strengths are far too low for them to be important for the cosmic evolution. The weak and the strong interaction, of course, have microscopic range and must thus be unimportant for cosmology as a whole.

- The best current theory of gravity is Einstein's theory of general relativity, which relates the geometry of a four-dimensional space-time manifold to its material and energy content. Cosmological models must thus be constructed as solutions of Einstein's field equations.

- Symmetry assumptions greatly simplify this process. Guided by observations to be specified later, we assume that the Universe appears approximately identically in all directions of observation, in other words, it is assumed to be isotropic on average. While this assumption is obviously incorrect in our cosmological neighbourhood, it holds with increasing precision if observations are averaged on increasingly large scales.

- Strictly speaking, the assumption of isotropy can only be valid in a prefered reference frame which is at rest with respect to the mean cosmic motion. The motion of the Earth within this rest frame must be subtracted before any observation can be expected to appear isotropic.

- The second assumption holds that the Universe should appear equally isotropic about *any* of its points. Then, it is homogeneous. Searching for isotropic and homogeneous solutions for Einstein's field equations leads uniquely to line element of the Robertson-Walker metric,

  $$\mathrm{d}s^2 = -c^2\mathrm{d}t^2 + a^2(t)\left[\frac{\mathrm{d}r^2}{1-kr^2} + r^2\left(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\phi^2\right)\right] , \quad (2.1)$$

  in which r is a radial coordinate, k is a parameter quantifying the curvature, and the *scale factor* $a(t)$ isotropically stretches or shrinks the three-dimensional spatial sections of the four-dimensional space-time; the scale factor is commonly normalised such that $a_0 = 1$ at the present time;

- as usual, the line element $\mathrm{d}s$ gives the proper time measured by an observer who moves by $(\mathrm{d}r, r\mathrm{d}\theta, r\sin\theta\mathrm{d}\phi)$ within the *coordinate time* interval $\mathrm{d}t$; for light, in particular, $\mathrm{d}s = 0$;

- coordinates can always be scaled such that the curvature parameter k is either zero or ±1;

- by a suitable transformation of the radial coordinate r, we can rewrite the metric in the form

  $$\mathrm{d}s^2 = -c^2\mathrm{d}t^2 + a^2(t)\left[\mathrm{d}w^2 + f_k^2(w)\left(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\phi^2\right)\right] , \quad (2.2)$$

  where the radial function $f_k(w)$ is given by

  $$f_k(w) = \begin{cases} \sin(w) & (k=1) \\ w & (k=0) \\ \sinh(w) & (k=-1) \end{cases} ; \quad (2.3)$$

sometimes one or the other form of the metric is more convenient;

2.4.2 Redshift and expansion

- the changing scale of the Universe gives rise to the cosmological redshift z; the wavelength of light from a distant source seen by an observer changes by the same amount as the Universe changes its scale while the light is travelling; thus, if λ_e and λ_o are the emitted and observed wavelengths, respectively, they are given by

$$\frac{\lambda_o}{\lambda_e} = \frac{a_0}{a} = \frac{1}{a} , \tag{2.4}$$

where a is the scale factor at the time of emission and a_0 is normalised to unity; the *relative* wavelength change is the redshift,

$$z \equiv \frac{\lambda_o - \lambda_e}{\lambda_e} = \frac{1}{a} - 1 , \tag{2.5}$$

and thus

$$1 + z = \frac{1}{a} , \quad a = \frac{1}{1+z} ; \tag{2.6}$$

- when inserted into Einstein's field equations, two ordinary differential equations for the scale factor $a(t)$ result; when combined, they can be brought into the form

$$\begin{aligned} H(a)^2 &= H_0^2 \left[\frac{\Omega_{\mathrm{m},0}}{a^3} + \frac{\Omega_{\mathrm{r},0}}{a^4} + \Omega_\Lambda + \frac{1 - \Omega_{\mathrm{m},0} - \Omega_{\mathrm{r},0} - \Omega_\Lambda}{a^2} \right] \\ &\equiv H_0^2 E^2(a) ; \end{aligned} \tag{2.7}$$

this is Friedmann's equation, in which the relative expansion rate $\dot{a}/a \equiv H(a)$ is replaced by the *Hubble function* whose present value is the *Hubble constant*, and the matter-energy content is described by the three density parameters $\Omega_{\mathrm{r},0}$, $\Omega_{\mathrm{m},0}$ and $\Omega_{\Lambda,0}$;

- the dimension-less parameters $\Omega_{\mathrm{m},0}$ and $\Omega_{\mathrm{r},0}$ describe the densities of matter and radiation in units of the critical density

$$\rho_{\mathrm{cr},0} \equiv \frac{3H_0^2}{8\pi G} ; \tag{2.8}$$

matter and radiation are distinguished by their pressure; for matter, the pressure p is neglected because it is very small compared to the energy density ρc^2, while radiation is characterised by $p = \rho c^2/3$;

- a Robertson-Walker metric whose scale factor satisfies Friedmann's equation is called a Friedmann-Lemaître-Robertson-Walker metric; the cosmological standard model asserts that the Universe at large is described by such a metric, and is thus characterised by the four parameters $\Omega_{\mathrm{m},0}$, $\Omega_{\mathrm{r},0}$, Ω_Λ and H_0;

- since the critical density evolves in time, so do the density parameters; their evolution is given by

$$\Omega_{\rm m}(a) = \frac{\Omega_{\rm m,0}}{a + \Omega_{\rm m,0}(1-a) + \Omega_{\Lambda,0}(a^3 - a)} \tag{2.9}$$

for the matter-density parameter and

$$\Omega_{\Lambda}(a) = \frac{\Omega_{\Lambda,0}a^3}{a + \Omega_{\rm m,0}(1-a) + \Omega_{\Lambda,0}(a^3 - a)} \tag{2.10}$$

for the cosmological constant; in particular, these two equations show that $\Omega_{\rm m}(a) \to 1$ and $\Omega_{\Lambda}(a) \to 0$ for $a \to 0$, independent of their present values, and that $\Omega_{\rm m}(a)+\Omega_{\Lambda}(a) = 1$ if $\Omega_{\rm m,0}+\Omega_{\Lambda,0} = 1$ today;

- this lecture is devoted to answering two essential questions: (1) What are the values of the parameters defining characterising Friedmann's equation? (2) How can we understand the *deviations* of the real universe from a purely homogeneous and isotropic space-time?

2.4.3 Age and distances

- since Friedmann's equation gives the relative expansion rate $\dot{a}/a$, we can use it to infer the age of the Universe,

$$t = \int_0^t {\rm d}t' = \int_0^1 \frac{{\rm d}a}{\dot{a}} = \int_0^1 \frac{{\rm d}a}{aH(a)} = \frac{1}{H_0}\int_0^1 \frac{{\rm d}a}{aE(a)}\,, \tag{2.11}$$

which illustrates that the age scale is the inverse Hubble constant H_0^{-1}; a simple example is given by the Einstein-de Sitter model, which (unrealistically, as we shall see later) assumes $\Omega_{\rm m,0} = 1$, $\Omega_{\rm r,0} = 0$ and $\Omega_{\Lambda} = 0$; then, $E(a) = a^{-3/2}$ and

$$t = \frac{1}{H_0}\int_0^1 \sqrt{a}{\rm d}a = \frac{2}{3H_0}\;; \tag{2.12}$$

- distances can be defined in many ways which typically lead to different expressions; we summarise the most common definitions here; the *proper distance* $D_{\rm prop}$ is the distance measured by the light-travel time, thus

$${\rm d}D_{\rm prop} = c{\rm d}t \quad \Rightarrow \quad D_{\rm prop} = \frac{c}{H_0}\int \frac{{\rm d}a}{aE(a)}\,, \tag{2.13}$$

where the integral has to be evaluated between the scale factors of emission and observation of the light signal;

- the *comoving distance* $D_{\rm com}$ is simply defined as the distance measured along a radial light ray ignoring changes in the scale factor, thus $\mathrm{d}D_{\rm com} = \mathrm{d}w$; since light rays propagate with zero proper time, $\mathrm{d}s = 0$, which gives

$$\mathrm{d}D_{\rm com} = \mathrm{d}w = \frac{c\mathrm{d}t}{a} = \frac{c}{H_0}\int\frac{\mathrm{d}a}{a\dot{a}} = \frac{c}{H_0}\int\frac{\mathrm{d}a}{a^2E(a)} \; ; \qquad (2.14)$$

- the *angular-diameter distance* $D_{\rm ang}$ is defined such that the same relation as in Euclidean space holds between the physical size of an object and its angular size; it turns out to be

$$D_{\rm ang}(a) = af_K[w(a)] = af_K[D_{\rm com}(a)] \; , \qquad (2.15)$$

  where $f_K(w)$ is given by (2.3);

- the *luminosity distance* $D_{\rm lum}$ is analogously defined to reproduce the Euclidean relation between the luminosity of an object and its observed flux; this gives

$$D_{\rm lum}(a) = \frac{D_{\rm ang}(a)}{a^2} = \frac{f_K[w(a)]}{a} = \frac{f_K[D_{\rm com}(a)]}{a} \; , \qquad (2.16)$$

- these distance measures can vastly differ at scale factors $a \ll 1$; for small distances, i.e. for $a \sim 1$, they all reproduce the linear relation

$$D(z) = \frac{cz}{H_0} \; . \qquad (2.17)$$

- since time is finite in a universe with Big Bang, any particle can only be influenced by, and can only influence, events within a finite region; such regions are called *horizons*; several different definitions of horizons exist; they are typically characterised by some speed, e.g. the light speed, times the inverse Hubble function which sets the time scale;

2.4.4 The radiation-dominated phase

- it is an empirical fact that the Universe is expanding; earlier in time, therefore, the scale factor must have been smaller than today, $a < 1$; in principle, it is possible for Friedmann models that they had a finite minimum size at a finite time in the past and thus never reached a vanishing radius, $a = 0$; however, it turns out that a few crucial observational results rule out such "bouncing" models; this implies that a Unniverse like ours which is expanding today must have started from $a = 0$ a finite time ago, in other words, there must have been a Big Bang;

- equation (2.7) shows that the radiation density increases like a^{-4} as the scale factor decreases, while the matter density increases with one power of a less; even though the radiation density is very much smaller today than the matter density, this means that there has been a period in the early evolution of the Universe in which radiation dominated the energy density; this *radiation-dominated era* is very important for several observational aspects of the cosmological standard model;

- since the radiation retains the Planckian spectrum which it acquired in the very early Universe in the intense interactions with charged particles, its energy density is fully characterised by its temperature T; since the energy density is both proportional to T^4 and a^{-4}, its temperature falls like $T \propto a^{-1}$;

2.5 Structures

2.5.1 Structure growth

- the hierarchy of cosmic structures is assumed to have grown from primordial seed fluctuations in the process of gravitational collapse: overdense regions attract material and grow; they are described by the *density contrast* δ, which is the density fluctuation relative to the mean density $\bar{\rho}$,

$$\delta \equiv \frac{\rho - \bar{\rho}}{\bar{\rho}} \; ; \tag{2.18}$$

- linear perturbation theory shows that the density contrast δ is described by the second-order differential equation

$$\ddot{\delta} + 2H\dot{\delta} - 4\pi G\bar{\rho}\delta = 0 \tag{2.19}$$

  if the dark matter is cold, i.e. if its constituens move with negligible velocities; notice that this is an oscillator equation with an imaginary frequency and a characteristic time scale $(4\pi G\bar{\rho})^{-1/2}$, and a damping term $2H\dot{\delta}$ which shows that the cosmic expansion slows down the gravitational instability;

- equation (2.19) has two solutions, a growing and a decaying mode; while the latter is irrelevant for structure growth, the growing mode is described by the *growth factor* $D_+(a)$, defined such that the density contrast at the scale factor a is related to an initial density contrast δ_i by $\delta(a) = D_+(a)\delta_\mathrm{i}$; in most cases of practical relevance, the growth factor is accurately described by the fitting