Spiraling close-packers of 3D space

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Abstract

This paper presents a new class of three-dimensional mathematical objects that close-pack to fill space. These so-called Tordated Sinusoids ([N]TS) are based on rotational variations of two-dimensional tesselations of equilateral triangles (which forms TTS) and of squares (QTS). The combination of sixfold Tordated Sinusoids (HTS) and TTS gives rise to a composite 3D lattice. It appears that the ruled surfaces of an [N]TS are linearly tordated and are circumscribed by N circular spirals. It also appears that the TTS, QTS and HTS can be described as a cube, Kepler's dodecahedron and a triacontahedron respectively with their rhombic surfaces parametrized in curvilinear coordinates.

1. From 2D tesselations to 3D close-packing

Two-dimensional tesselations of a plane are well-studied by both mathematicians, crystallographers and artists alike. The basic regular tesselations are ascribed to Plato[1]. There are only three regular polygons which can be used to tile a surface using only likes: triangles, squares and hexagons. Tilings using combinations of different regular polygons are called Archimedean[1]. The Dutch graphic artist M.C. Escher (1898-1972) used such tesselations to bring his parallel universe with all its bizarre inhabitants come to life[2].

The present and earlier work[3] shows how Platonic and Archimedean tesselations can be elegantly extended into the third dimension to form spiraling close-packers.

Figure 1.

Tesselations of rotating squares with two different phases α.

a.	b.	c.	d.	e.
α=180° α=0°	α=225° α=45°	α=270° α=90°	α=315° α=135°	α=360° α=180°
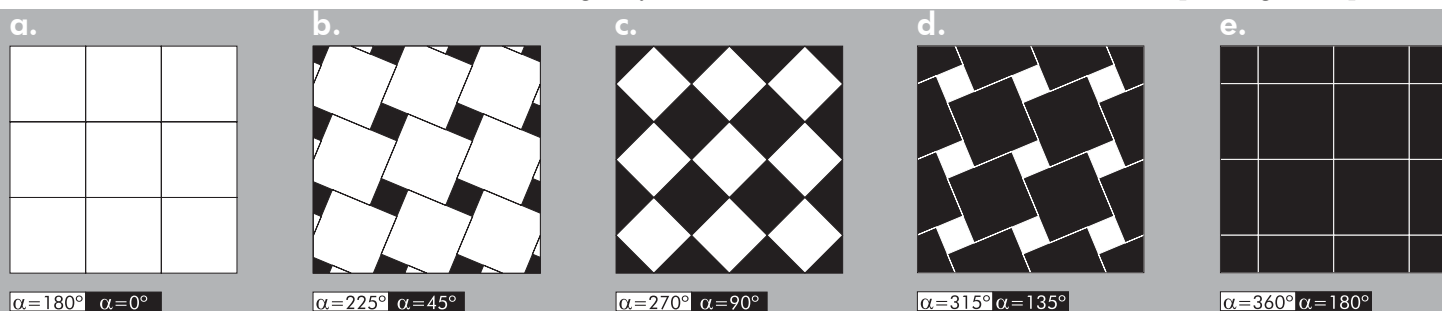
1.1. Consider the regular tesselation of white squares in figure 1a. By rotating each square around its center whilst decreasing its size, a second set of (black) squares appears in figure 1b at the locations of the vertices of the original white squares. By further rotating and scaling around their centers (conform the "snapshots" of figure 1c-d), the white and black squares generate a complete sequence of tesselations until the black squares form a regular Platonic tesselation like in figure 1e. By switching the colors here, we are basically back at the starting point, only with a spatial shift. But in fact the cycle is only halfway through here and it can be completed by rotating and scaling the black squares according to the same recipe until you are back at the situation of figure 1a.

This procedure gives us a clue that the evolution of a white square is identical to that of a black square, only with a 180° phase shift. By transforming the parametrization α of the sequence of figure 1 into a third spatial coordinate as in figure 2, it indeed appears that the white and black squares make up the same spiraling three-dimensional shape. Therefore this shape is an intrinsic close-packer of 3D space, just like a square is a close-packer of 2D space. For reasons that are revealed in the next section, these bodies are named "Tordated Sinusoids" of order N ([N]TS). The body with fourfold rotational symmetry based on squares is called "QuatTorSin" or just QTS. Of course these Tordated Sinusoids shape have twofold degeneracy: it can spiral both clockwise and counter-clockwise with equal packing properties. In crystallography the lattice structure of the QTS is called "body-centered cubic".

Figure 2.
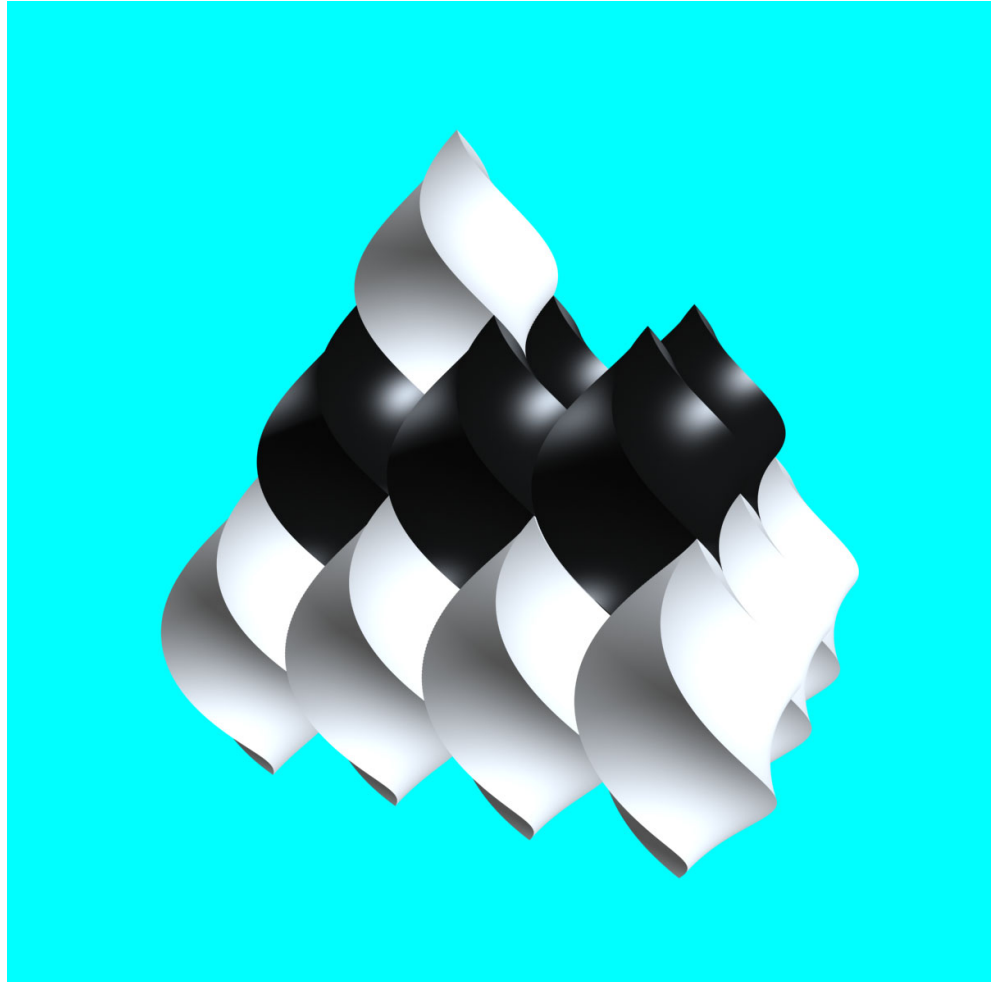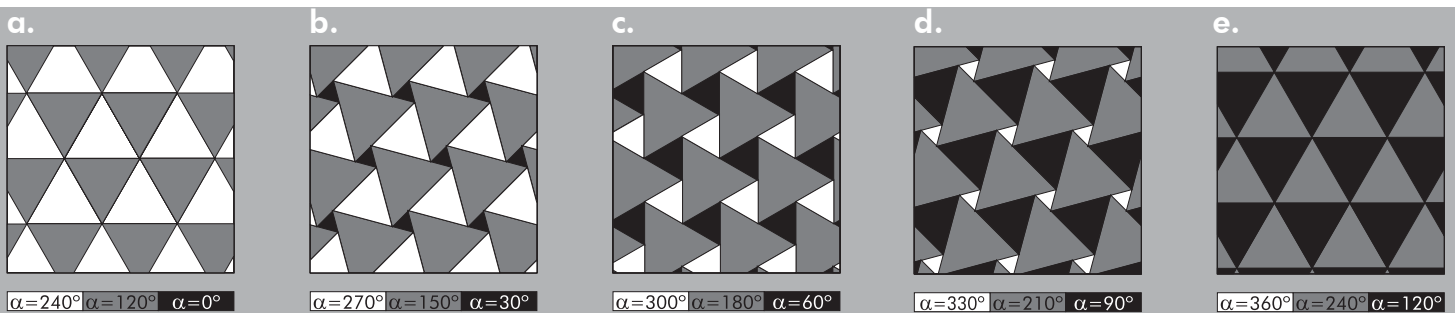
Three-dimensional close-packing of QTS.

Figure 3.

Tesselations of rotating equilateral triangles with three different phases α.

| a.	b.	c.	d.	e.
α=240° α=120° α=0°	α=270° α=150° α=30°	α=300° α=180° α=60°	α=330° α=210° α=90°	α=360° α=240° α=120°

1.2. The same trick is also possible with Platonic tilings of equilateral triangles as indicated in figure 3. Here, a third color (grey) is necessary to distinguish between the different sets of triangles. It is obvious that the three different sets have a phase shift of 120°.

Again making the step to 3D space, these sequences result in spiraling close-packers with threefold rotational symmetry, called "TriTorSins" (TTS). They are to be stacked in a "cubic" fashion as shown in figure 4.

Figure 4.

Three-dimensional close-packing of TTS.
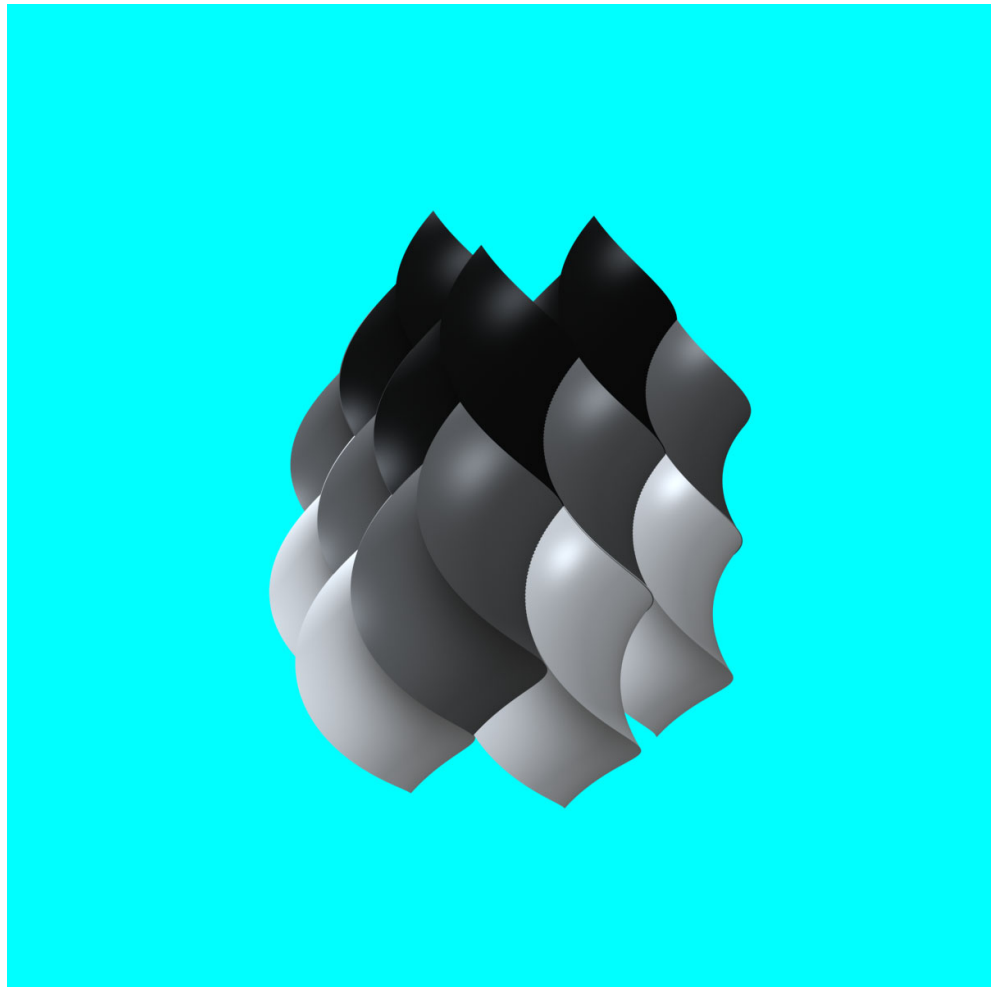
1.3. By trying out the exact same procedure of rotating and scaling for the Platonic tesselation of hexagons, the gaps turn out to be triangular instead of hexagonal like the orginal tesselations (figure 5). Apparently we need to resort to Archimedean tesselations for the intermediate tilings. Note that the triangles evolve just like in figure 3, but here there is only one single phase. Yet, there are two sets of triangles, rotated at 60° around their axes with respect to each other.

The combination of spiraling hexagons and the bodies of figure 4 give rise to a composite close-packing of "HexaTorSins" (HTS) and TTS in a 1:2 ratio as depicted in figure 6.

Figure 5.

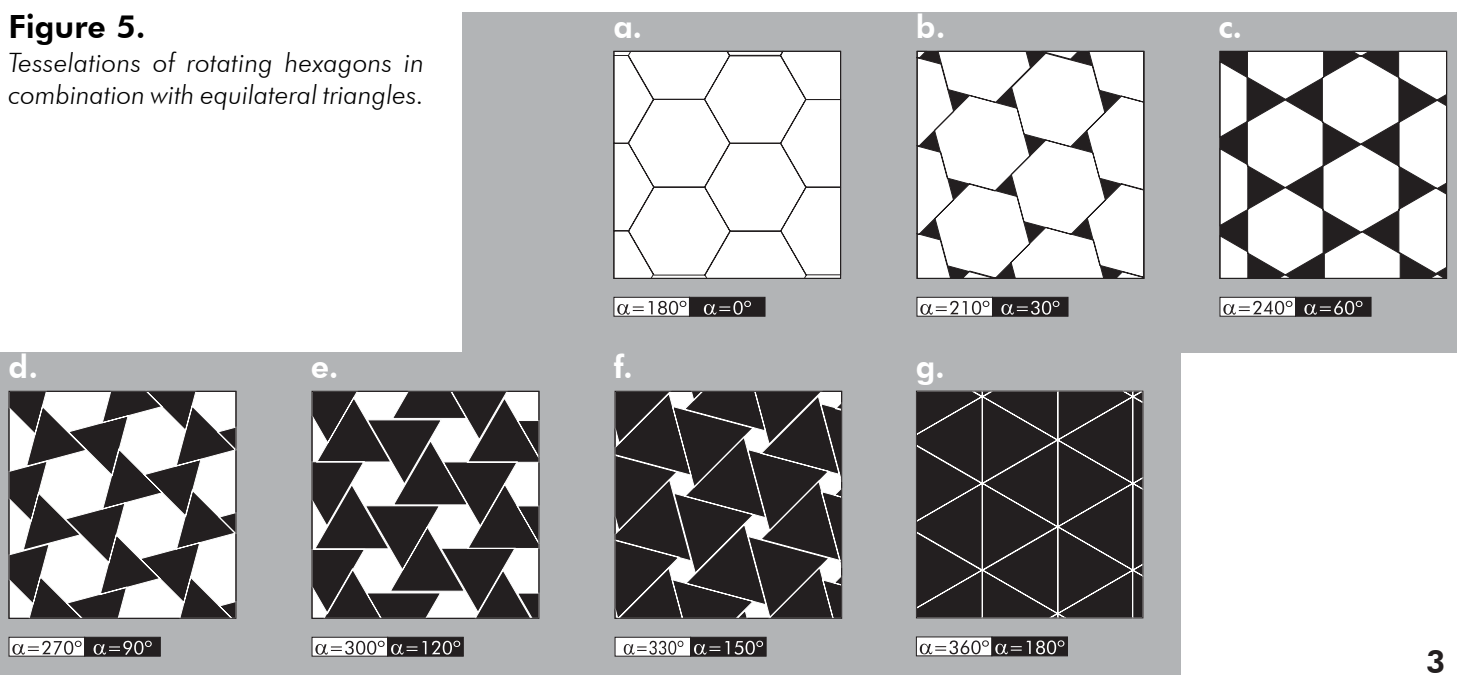*Tesselations of rotating hexagons in combination with equilateral triangles.*

a.
α=180° α=0°

b.
α=210° α=30°

c.
α=240° α=60°

d.
α=270° α=90°

e.
α=300° α=120°

f.
α=330° α=150°

g.
α=360° α=180°

Figure 6.

Three-dimensional close-packing of HTS and TTS.

1.4. The shapes and close-packing properties of the [N]TS are very similar to the Spirallohedra discovered and described by Towle [4]. These Spirallohedra are obtained by transformations of polar zonohedra and are therefore covered by many rhombic faces, whereas the N surfaces of an [N]TS are fully double-curved. In fact, the [N]TS are the limiting cases of the k-armed Spirallohedra when the number of faces goes to infinity. Therefore the horizontal cross-sections of a Spirallohedron generally will not deliver regular polygons. Nevertheless it is intriguing to see how two totally different approaches deliver similar objects.

2. Mathematical description of the ruled surfaces

Several methods can be employed to mathematically describe the ruled surfaces of an [N]TS.

2.1. The most intuitive way to describe the surface of an [N]TS is just to look at the tilings of figures 1, 3 and 5 as horizontal cross-sections of close-packing QTS, TTS and HTS/TTS respectively. By following the evolution of one square as in figure 7a it is possible to identify a limited number of cardinal grid-points that govern the rotation and scale. In three dimensions these points form vertical axes that describe the entire close-packing.

First of all, it is clear that the squares rotate around their center axes $A_{b,w}$. To enable continuity of the pattern, the edges (or the continuation thereof) have to cross fixed points B. These points B therefore form vertical straight lines on the tordated surface of a QTS. After a closer study of the patterns of figure 1 it appears that the vertices of the squares are all located on perfect circles after completion of a cycle. The centers of these circles are tagged with C. The circles intersect at A and B points. These circles are parametrized linearly (completing double the circular angle as compared to the rotation of the squares) and form either right- or left-handed screwing spirals in 3D space.

Figure 7b-c shows that also a TTS and a HTS are described with these A, B and C grid-points. It is concluded that all [N]TS only principally differ in the number of ruled surfaces N. But a study of Platonic and Archimedean tilings shows that only the bodies based on triangles (N=3), squares (N=4) and hexagons (N=6) are suitable for 3D close-packing.

Figure 7a-c.
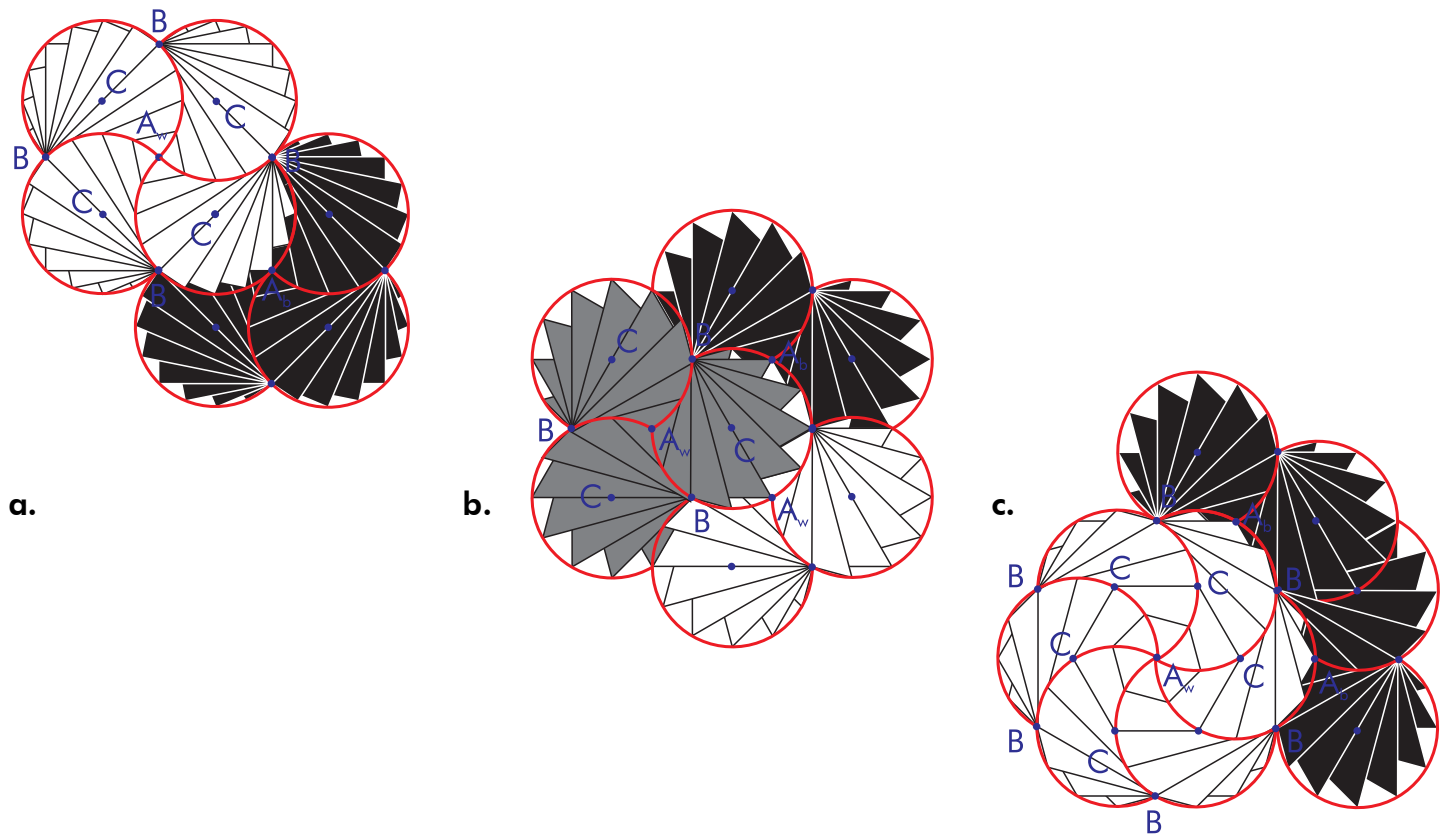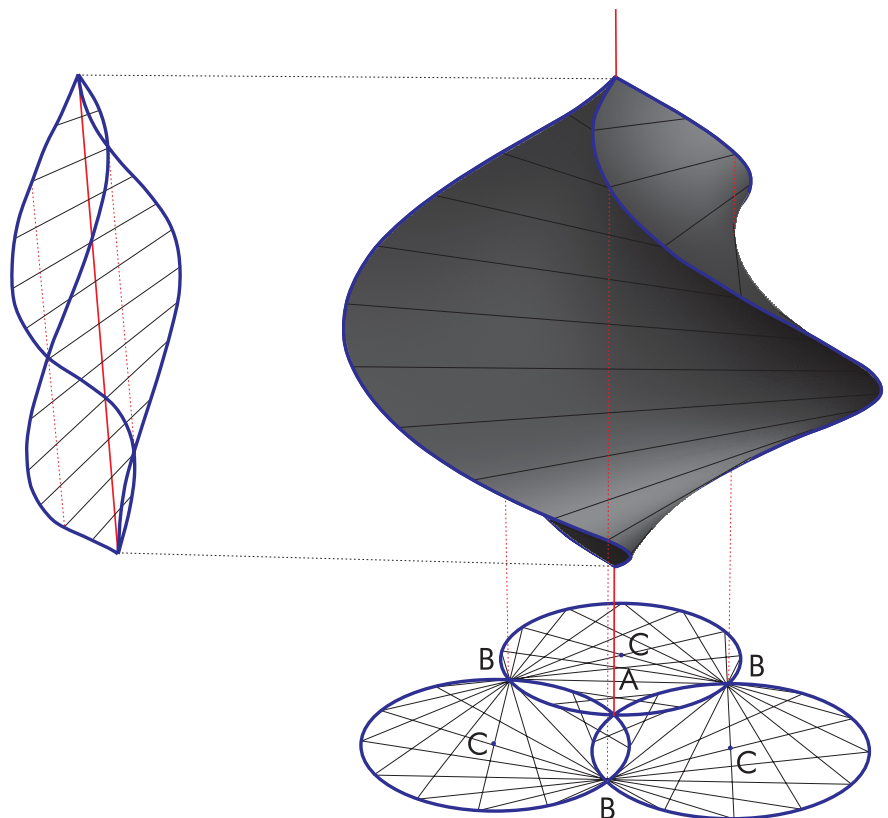*Cardinal axes of a QTS, TTS and HTS.*

a.

b.

c.

Figure 8 shows how the surface of an [N]TS can be built up from horizontal rules governed by the vertical axes A, B and C.

Figure 8.
The surface of a TTS as a horizontally ruled surface.

2.2. If the parametrization of the rotating polygons of figures 1, 3 and 5 is called α, which cycles through 360° for creating one [N]TS, it can be proven that the angle of one edge of the polygon with respect to the angle at α↓0 is proportional to ½α. In other words, the tordation of the surface ruling around the B axis is linear and completes 180° over the full length.

Furthermore, with the description of the circular orbits of the vertices the length of the edge of such a polygon can be shown to be proportional to sin(½α). This statement can easily be proven by taking the total surface of a white and a black square in figure 1, which of course needs to be constant. With implementing their phase relation of 180°, that surface is proportional to sin²(½α)+sin²(½(α+180°))=sin²(½α)+cos²(½α)=1. Equivalently for the triangles of figure 3, sin²(½(α-120°))+sin²(½α)+sin²(½(α+120°))=3/2.

This gives us a second recipe to construct an [N]TS as illustrated in figure 9. First create an N-fold "sinusoid" by stacking the N-gons over the height range 0<α<360° with their sizes proportional to sin(½α). Subsequently, tordate that body linearly over 180° from top to bottom either clockwise or counter-clockwise. It can be proven that this operation results in the exact close-packing properties of the TTS, QTS and HTS.

It is clear that this description of tordating sinusoids is what gave the [N]TS their name. Also the description of the vertices with circular spirals is of course governed by sine functions.
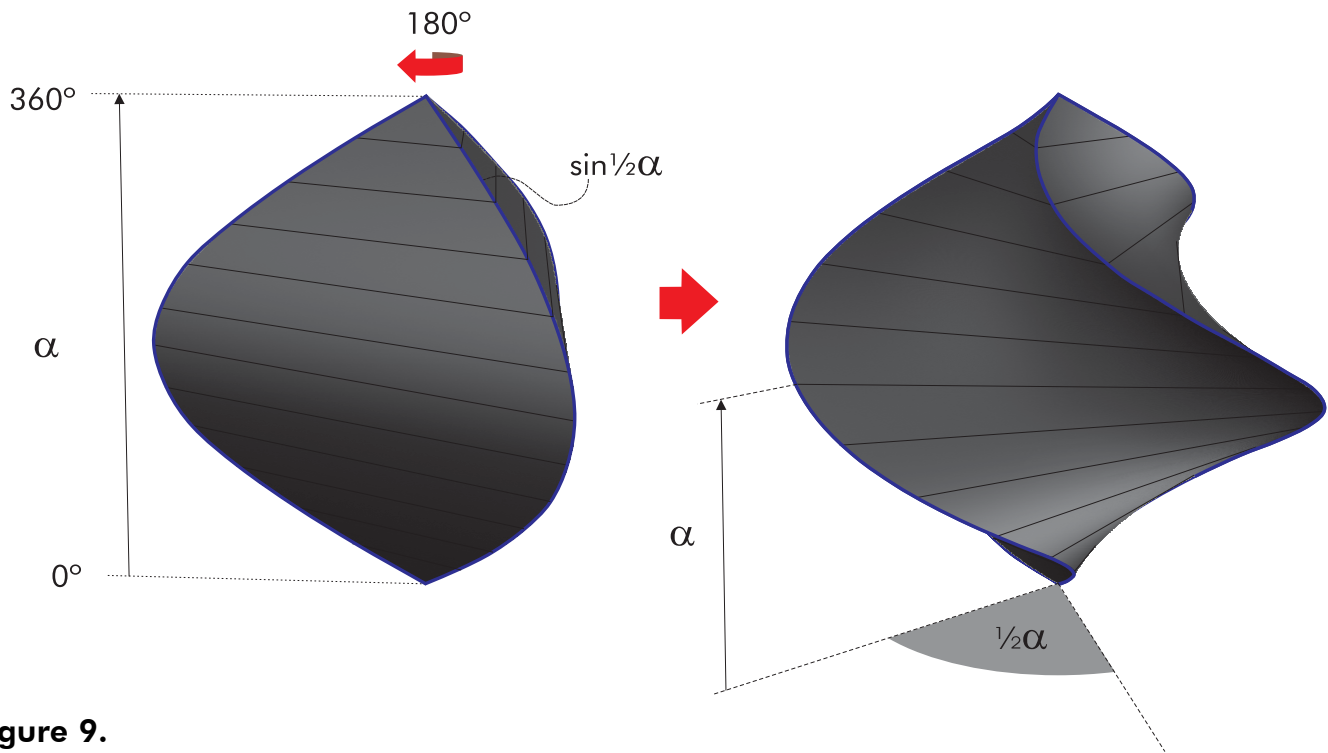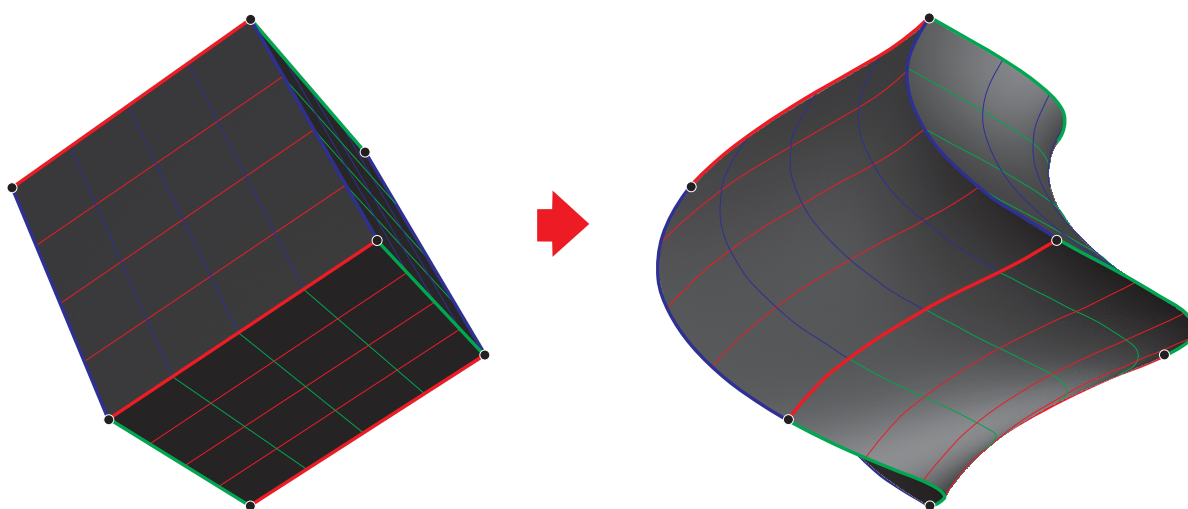

Figure 9.
The surface of a TTS as obtained by linearly tordating a threefold sinusoid over 180°.

2.3. The lattice structure of the close-packing of TTS and QTS is exactly the same as the close-packing of cubes and Kepler's dodecahedron respectively. These rhombohedra are known to be the only regular bodies that can close-pack to fill 3D space[1]. When the dimensions of the TTS and the QTS are scaled in a way that their spiraling edges are "aligned" with the edges of their accompanying rhombohedron like in figure 10, it can be easily shown that both bodies have exactly the same volume.

Figure 10.

*The surface of a TTS as obtained by a
curvilinear transformation of the faces
of a cube.*

These two clues make clear that a TTS is in fact equal to a cube with its faces made up as a linear combination of the spiraling edges as shown in figure 10. Similarly, the QTS can be described as Kepler's rhombic dodecahedron with its rhombic faces in curvilinear coordinates. Note that this involves a symmetry breaking in all the axes of the cube, exept for rotational symmetry. It can be proven that these transformations will exactly produce the horizontal rulings and the vertical B lines of section 2.1.

The rhombic analogon of a HTS is a triacontahedron (30-faced polyhedron). The same space-filling properties as with the HTS and the TTS can only be accomplished with the use of the lowest order 3-armed Spirallohedra (24 faces).

The transformation from a cube to a right- or left-handed TTS along any body diagonal can be parametrized by the direction and amount of curvature of the cube's edges. This leads to a cyclic body in yet a higher dimension of which the cube and all the TTS variations are 3D cross-sections.

3. Tesselating sculptures

The range of mathematical bodies [N]TS as described above form a surprising set of spiraling close-packers for N=3, 4 and 3/6. Almost in contradiction to their very organic shapes they can be used as "bricks" to build intricate three-dimensional structures. This creates many new possibilities for sculptors, computer artists and architects. With the recipes of section 2 the shapes are relatively easily created.

The spiraling shape is fascinating in itself with its mathematical elegance. Of course it reflects nature on all scales, from DNA to spiral galaxies. With the space-filling properties decribed in this paper they even encompass infinity.

References

[1]   Coxeter, H.S.M. "*Regular Polytopes*", 3rd ed., 1973, Dover Publications, New York
[2]   Schattschneider, D. "*M.C. Escher: visions of symmetry*", 2nd ed., 2004, Harry N. Abrams
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[4]   Towle, R. "*Rhombic spirallohedra*", Symmetry: Culture and Science, 11, 1-4, p.293-306